18.704 Supplementary Notes
February 4, 2005

Orbits and counting formulas

The point of looking at group actions is to understand better the sets on which the groups act, or the groups themselves. For us in this seminar, the most fundamental kind of understanding is counting. These notes describe a basic counting formula for group actions. As usual, we need a definition or two to start.

Definition 1. Suppose that the group G acts on the set X, and that $x \in X$. The *orbit of* x consists of all points that can be reached from x using the group action:

$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

The action of G on X is called *transitive* if X consists of exactly one orbit.

If we think of G as a symmetry group of X, then sometimes it's useful to think of the orbit as consisting of all the points that are indistinguishable from X up to symmetry. For example, suppose X is an equilateral triangle with vertices A, B, and C, and G is the six-element group of symmetries we looked at in the seminar. The orbit of A consists of the three vertices:

$$G \cdot A = \{A, B, C\}.$$

The vertices are indistinguishable up to symmetry. Suppose P is the point one-third of the way along the edge from A to B. You should convince yourself that the orbit of P consists of six points (and figure out what those six points are).

Example 1. Suppose that $X = \mathbb{R}^2$, and G is the group of invertible 2×2 matrices as in Example 3 from the Groups notes. If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g_T \cdot (0,0) = (0,0), \qquad g_T \cdot (1,0) = (a,c).$$

The first formula shows that the orbit of $(0,0)$ is just the single point $(0,0)$. The second shows that the orbit of $(1,0)$ consists of all vectors that can appear as the first column of an invertible matrix. It's a fact from linear algebra that *any* non-zero vector can be the first column of an invertible matrix. (Do you remember how to prove that?) Therefore

$$G \cdot (1,0) = \mathbb{R}^2 \setminus \{(0,0)\}.$$

Definition 2. Suppose that the group G acts on the set X, and that $x \in X$. The *isotropy group at* x is the subgroup of G consisting of the elements that do not move x:

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

(Why is this a subgroup?)

The isotropy group G_A of the vertex A in the triangle symmetry group consists of all symmetries fixing the vertex A: in the notation of the Tables handout,

$$G_A = \{id, refl_A\}.$$

Example 2. In the setting of Example 1, $G_{(0,0)} = G$, and

$$G_{(1,0)} = \left\{ g_T \mid T = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}, \quad d \neq 0 \right\}.$$

Here is what you might regard as the main theorem for this seminar.

Theorem 1. *Suppose that the finite group G acts on the set X, and that $x \in X$. Then the cardinality of G is the product of the cardinality of the orbit of x and the cardinality of the isotropy group at x. Writing $|S|$ for the cardinality of a finite set S, this is*

$$|G| = |G \cdot x|\,|G_x|.$$

Proof. The idea of the proof is to define a map α going from the group G to the orbit $G \cdot x$. (The letter α stands for "action.") I will show that for every point $y \in G \cdot x$, the preimage $\alpha^{-1}(y)$ (a subset of G) has exactly $|G_x|$ points. That is, α makes a correspondence between elements of G and points of $G \cdot x$, in which every point of $G \cdot x$ corresponds to $|G_x|$ group elements. The conclusion of the theorem follows.

Here are the details. The map α is

$$\alpha\colon G \to G \cdot x, \qquad \alpha(g) = g \cdot x.$$

So fix a point $y \in G \cdot x$; we want to understand the preimage of y. What it means to say that y is in the orbit of x is that there is *some* element $g_0 \in G$ such that $g_0 \cdot x = y$. Therefore $\alpha(g_0) = y$, so g_0 is an element of $\alpha^{-1}(y)$. We want to understand all the other elements in $\alpha^{-1}(y)$. So suppose g_1 is another; that is, that

$$\alpha(g_1) = g_1 \cdot x = y = g_0 \cdot x.$$

Applying g_0^{-1} to both sides of the equation $g_1 \cdot x = g_0 \cdot x$, we get on the right

$$\begin{aligned} g_0^{-1} \cdot (g_0 \cdot x) &= (g_0^{-1} \cdot g_0) \cdot x \qquad \text{(property (1) of a group action)} \\ &= 1 \cdot x \qquad \text{(property of inverses in a group)} \\ &= x \qquad \text{(property (2) of a group action)} \end{aligned}$$

On the left side, a shorter calculation gives $(g_0^{-1} g_1) \cdot x$. From the equality of the two sides, we deduce that $(g_0^{-1} g_1) \cdot x = x$, which by Definition 2 amounts to

$$g_0^{-1} g_1 = h \in G_x.$$

Multiplying this equation by g_0 on the left gives

$$g_1 = g_0 h \qquad (h \in G_x).$$

All of these arguments are reversible, and it follows that any element of the form $g_0 h$ (with $h \in G_x$) belongs to $\alpha^{-1}(y)$.

Here is what we have proved. If g_0 is any element such that $g_0 \cdot x = y$, then

$$\alpha^{-1}(y) = \{g_0 h \mid h \in G_x\}.$$

We have therefore listed all the elements of $\alpha^{-1}(y)$ by using all the elements of G_x. It's easy to see that each element of $\alpha^{-1}(y)$ is listed exactly once: if $g_0 h_1 = g_0 h_2$, then multiplying both sides by g_0^{-1} on the left shows that $h_1 = h_2$. Therefore

$$|\alpha^{-1}(y)| = |G_x|,$$

as we wished to show. $\square$

This proof has a lot of mileage in it; the rest of the material in these notes is just mathematical decoration of it. Here is a definition to get started.

Definition 3. Suppose G is a group and H is a subgroup. A *left coset of H in G* is a subset of G of the form

$$aH = \{g \in G \mid g = ah, \quad \text{some } h \in H\}.$$

More specifically, the set aH is called the *left coset of H*. The *homogeneous space* G/H is by definition the set of all left cosets of H in G:

$$G/H = \{aH \mid a \in G\}.$$

The *natural action of G on G/H* is defined by

$$g \cdot (aH) = (ga) \cdot H = \{y \in G \mid y = gx, \quad \text{some } x \in aH\}.$$

The second formula in this definition shows that $g \cdot aH$ really depends only on the coset aH, and not on the particular element a chosen to represent it. You should check that this is really an action of G on the set G/H.

Theorem 2. *Suppose that the group G acts on the set X, and that H is a subgroup of G.*

(1) *Two cosets aH and bH have non-empty intersection if and only if they are the same. This happens if and only if $b^{-1}a \in H$.*
(2) *The action of G on G/H is transitive. The isotropy group of the action at the identity coset $x = 1 \cdot H$ is equal to H.*
(3) *Suppose that $x \in X$, and that the isotropy group G_x is equal to H. Then there is a one-to-one correspondence*

$$\beta \colon G/H \to G \cdot x, \qquad \beta(aH) = a \cdot x$$

between the set of cosets of H in G, and the orbit of x in X. This one-to-one correspondence respects the actions of G:

$$g \cdot \beta(aH) = \beta(g \cdot aH).$$

The proof is essentially the same as the proof of Theorem 1: you should make sure that you understand how to fill in the details.

This theorem says that transitive actions of G are essentially the same as subgroups of G.