Clifford Algebras as Filtered Algebras

Yaim Cooper

May 11, 2005

1 The Tensor Algebra

Definition. *An algebra is a vector space V over a field F with a multiplication. The multiplication must be distributive and, for every $f \in F$ and $x, y \in V$ must satisfy $f(xy) = (fx)y = x(fy)$.*

Our favorite example of an algebra is $M_n(F)$ — n by n matrices over F. It has the vector space structure over $\mathbb{F}$ of $\mathbb{F}^{n^2}$, and the usual matrix multiplication makes it an algebra over F.

There are very general ways of defining the tensor product of two vector spaces V and W, but here I will stick to a very concrete basis-dependent one, for finite dimensional vector spaces V and W with bases $\{e_1, ...e_n\}$ and $\{f_1, ...f_m\}$, respectively.

Definition. *The tensor product of V and W is the vector space $V \bigotimes W$ is spanned by elements of the form $v \otimes w$, where the following rules are satisfied, with a and element of the field $\mathbb{F}$:*

1. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
2. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$
3. $a(v \otimes w) = (av) \otimes w = v \otimes (aw)$

Aside: For those who know about dual spaces, $W \bigotimes V^*$ is the space of all linear maps from V to W. So our most familiar example of an algebra — $n \times n$ matrices — is in fact the tensor product $V \bigotimes V^*$. One of the cannonical ways of defining the tensor product $V \bigotimes W$ is in fact as the space of linear maps from V to W^*.

We can certainly take the tensor product of a vector space V with itself. We can also do this as many times as we feel like.

Definition. *For $k \geq 1$, define*
$\mathrm{T}^k(V) = V \bigotimes V ... \bigotimes V$ *(k factors)*
and $\mathrm{T}^0(V) = \mathbb{F}$

Now, we define the tensor algebra as

Definition. $\mathrm{T}(V) = \bigoplus_{k=0}^{\infty} \mathrm{T}^k(V)$

2 The Exterior Algebra

The tensor algebra is a ring - it has a multiplication which is distributive over the vector space addition. In this ring is an ideal $A(V)$ generated by all elements of the form $v \otimes v$, $v \in V$.

Definition. *The quotient of* $\mathrm{T}(V)$ *by the ideal* $\mathrm{A}(V)$ *is called the exterior algebra of* V*, and denoted by* $\bigwedge V$*.*

Now we define the concept of a graded algebra.

Definition. *An algebra* A *is graded if it is the direct sum of subspaces* $A = A_0 \bigoplus A_1 \bigoplus ...$ *such that* $A_i A_j \subseteq A_{i+j}$ *for all* $i, j \geq 0$*. The elements of* A_k *are said to be homogenous of degree k.*

Both the tensor algebra and exterior algebras defined thus far are graded algebras, the graded k^{th} piece in both cases being given by k-products of vectors.

3 The Clifford Algebra

The definition of a Clifford algebra is parallel to that of the exterior algebra. This time, we construct the ideal $D(V)$ generated by all elements of the form $v \otimes v - |v|^2$.

Definition. *The Clifford algebra of* V $\mathrm{C}(V)$ *is the quotient* $\mathrm{T}(V)/\mathrm{D}(V)$*.*

The Clifford algebra is no longer graded. This is because now when one multiplies two elements together, the resulting term may drop in degree without becoming zero, by the relation $v \otimes v = |v|^2$. For example, take v itself, of degree 1. $v \otimes v$ reduces to $|v|^2$, of degree zero, rather than a homogenous term of degree 2. Still, the product of a degree i term and degree j term can still give a result of degree at most $i + j$. So the algebra is no longer graded, but it does have a weaker property, that of being a filtered algebra.

Definition. *A filtration of an algebra* A *is a sequence of subspaces* $A_0 \subseteq A_1 \subseteq ...$ *such that* $A_i A_j \subseteq A_{i+j}$ *for all* $i, j \geq 0$*.*

This property is weaker than being graded in that we no longer have a direct sum decomposition, but rather a decomposition into a sequence of inclusive subspaces.

However, we can take the Clifford algebra and obtain from it a graded algebra again. We do so by defining the subspaces $gr(C(V)_p) = C(V)_p / C(V)_{p-1}$, and letting $gr(C(V)) = C(V)_0 \bigoplus gr(C(V)_1) \bigoplus gr(C(V)_2)...$

We end with the remarkable theorem,

Theorem 1. *$gr(\mathrm{C}(V)) \cong \bigwedge V$.*