Clifford Algebras and Isomorphisms

Eitan Reich

eitan@mit.edu

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To begin our discussion of Clifford algebras, we will make several definitions. The first definition is that of an algebra, which is the basic structure upon which a Clifford algebra is built.

Definition Let F be a field. An Algebra A is a vector space over F with a multiplication that is distributive and satisfies that for every $f \in F$ and $x, y \in A$:

$$f(xy) = (fx)y = x(fy)$$

For this presentation we will assume that A is an associative algebra with a multiplicative identity 1_A. We will also be working with a nondegenerate quadratic space V over F with a quadratic form Q. A Clifford algebra is one in which the quadratic form on V is strongly related to the mulitplication of elements in the algebra. To make this property clear, we will define when an algebra is compatible with a quadratic form:

Definition An algebra A is compatible with a quadratic form Q over V if there exists a linear transformation $\varphi : V \to A$ such that for all $v \in V$ we have $\varphi(v)^2 = Q(v) \cdot 1_A$.

We will now move on to the subject of Clifford algebras. We have seen the Clifford algebra constructed using the tensor algebra. In this lecture we will define the Clifford algebra using the property of compatibility. So we will construct a Clifford algebra for a quadratic space V with a quadratic form Q and then look at the isomorphisms we can find among these Clifford algebras.

Definition A Clifford algebra C for a quadratic space V is one that is compatible with V (via $\sigma : V \to C$) and such that for any other algebra A that is compatible with V (via $\varphi : V \to A$), there is a unique homomorphism $\hat{\varphi} : C \to A$ such that the following diagram commutes:

$$V \xrightarrow{\sigma} C$$
$$\varphi \searrow \quad \downarrow \varepsilon$$
$$A$$

We may now ask how the Clifford algebras for a given quadratic space V are related. The following proposition will show that there really is only one Clifford algebra $C(V)$ for a given quadratic space.

Proposition 0.1 *The Clifford algebra C for a quadratic space V is unique up to an algebra isomorphism.*

Proof We are given a Clifford algebra C with a mapping $\sigma : V \to C$ as in the definition above. Assume there is another Clifford Algebra C_1 for V. Since C_1 is an algebra that is compatible with the quadratic form for V, we have another mapping $\varphi : V \to C_1$ as above. By the fact that C is a Clifford algebra, we have a unique homormorphism $\varepsilon : C \to C_1$ such that the following diagram commutes:

$$V \xrightarrow{\sigma} C$$
$$\varphi \searrow \quad \downarrow \varepsilon$$
$$C_1$$

By the fact that C_1 is a Clifford algebra, we also have a unique homomorphism $\varepsilon_1 : C_1 \to C$ such that the following diagram commutes:

$$V \xrightarrow{\varphi} C_1$$
$$\sigma \searrow \quad \downarrow \varepsilon_1$$
$$C$$

So we have $\varphi = \varepsilon\sigma$ and $\sigma = \varepsilon_1\varphi$ as equations of homomorphisms. From these we can deduce that $\varphi = \varepsilon\varepsilon_1\varphi$ and $\sigma = \varepsilon_1\varepsilon\sigma$. This gives us the following diagram:

$$V \xrightarrow{\sigma} C$$
$$\sigma \searrow \quad \downarrow \varepsilon_1\varepsilon$$
$$C$$

This diagram illustrates the unique homomorphism $\varepsilon_1\varepsilon$ from C to itself. But since we also have the identity map from C to itself which allows the diagram to commute, and the homomorphism must be unique, we know that $\varepsilon_1\varepsilon$ is the identity on C. Similarly we can show that $\varepsilon\varepsilon_1$ is the identity on C_1. This shows that ε and ε_1 are inverse isomorphisms and

thus C is isomorphic to C_1. Furthermore, we can see that the isomorphisms found here are unique, which will be useful later.

Definition An antiautomorphism of an algebra A is a map $f : A \to A$ that preserves addition and satisfies $f(xy) = f(y)f(x)$ for all $x, y \in A$.

Definition The opposite algebra A^{op} of an algebra A is the algebra defined by the same elements and operations except that multiplication is reversed: $x *_{(A^{op})} y = y *_{(A)} x$ for all $x, y \in A$.

We can see that an equivalent definition of an antiautomorphism is as an isomorphism from an algebra into its opposite algebra. This brings us to our last result.

Proposition 0.2 *Given the Clifford algebra C for a quadratic space V, there is a unique anti-automorphism $f : C \to C$ satisfying $f|V = 1_V$.*

Proof We are given a map $\sigma : V \to C$ and for any algebra A that is also compatible with V we are also given a map $\varphi : V \to A$. We can see that the opposite algebra A^{op} is also compatible with V via the same map φ because $\varphi(x)^2$ is the same whether computed in A or A^{op}. So there must be a unique homomorphism $\varepsilon : C \to A^{op}$ as in the diagram below:

$$
\begin{array}{ccc}
V & \xrightarrow{\sigma} & C \\
 & \searrow{\varphi} & \downarrow{\varepsilon} \\
 & & A^{op}
\end{array}
$$

We see that this diagram is preserved if we substitute C^{op} for C and A^{op} for A:

$$
\begin{array}{ccc}
V & \xrightarrow{\sigma} & C^{op} \\
 & \searrow{\varphi} & \downarrow{\varepsilon} \\
 & & A
\end{array}
$$

Since we could find a unique homomorphism $\varepsilon : C^{op} \to A$, C^{op} must be a Clifford algebra by definition. Therefore it must be isomorphic to C via a unique isomorphism f. Since C and C^{op} are both constructed from V using the same map φ, the isomorphism between the two Clifford algebras must be the identity on $\varphi(C)$ which we just refer to as 1_V. Therefore we have found a unique antiautomorphism of C which is the identity when restricted to the image of V.