Unitary representations of reductive groups 6–10

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What can we ask about representations?

Start with a reasonable category of representations.

Example: $\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$; Bernstein-Gelfand-Gelfand category $\mathcal{O}$ consists of $U(\mathfrak{g})$-modules V subject to

1. fin gen: $\exists V_0 \subset V,\ \dim V_0 < \infty,\ U(\mathfrak{g})V_0 = V$.
2. $\mathfrak{b}$-locally finite: $\forall v \in V,\ \dim U(\mathfrak{b})v < \infty$.
3. $\mathfrak{h}$-semisimple: $V = \sum_{\gamma \in \mathfrak{h}^*} V(\gamma)$.

Want precise information about reps in the category.

Example: V in category $\mathcal{O}$

1. $\dim V(\gamma)$ is *almost polynomial* as function of γ.
2. $\mathrm{Ass}(V)$ is int comb of B-stable irr cones in $(\mathfrak{g}/\mathfrak{b})^*$.
3. V has a *formal character* $\left[\sum_{\lambda \in \mathfrak{h}^*} a_V(\lambda) e^\lambda\right] / \Delta$.

Want construction/classification of reps in the category.

Example: $\lambda \in \mathfrak{h}^* \rightsquigarrow I_\lambda =_{\mathrm{def}} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda =$ *Verma module*.

1. (SUBQUOTIENT THM): for every irr $J \in \mathcal{O}$ $\exists \lambda$ dominant with J comp factor of I_λ.
2. (STRUCTURE THM): $\exists \mathbb{C}_\lambda \hookrightarrow I_\lambda^{\mathfrak{n}}$.
3. (LANGLANDS THM): I_λ has *unique* irr quo J_λ; satisfies $\mathbb{C}_\lambda \hookrightarrow J_\lambda^{\mathfrak{n}}$.
4. Each irr in $\mathcal{O}$ is J_λ for unique $\lambda \in \mathfrak{h}^*$.

How do you do that?

$\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ roots, Δ^+ roots in $\mathfrak{n}$.

Introduce partial order on $\mathfrak{h}^*$:

$$\mu' \leq \mu \iff \mu' \in \mu - \mathbb{N}\Delta^+ :$$

that is, that $\mu' = \mu - \sum_{\alpha \in \Delta^+} n_\alpha \alpha$, with $n_\alpha \in \mathbb{N}$.

Proposition

Suppose $V \in \mathcal{O}$.

1. $\exists \{\lambda_1, \ldots, \lambda_r\} \subset \mathfrak{h}^*$ *so* $V(\mu') \neq 0 \implies \exists i, \mu' \leq \lambda_i$.
2. *If* $V \neq 0, \exists$ maximal $\mu \in \mathfrak{h}^*$ *subject to* $V(\mu) \neq 0$.
3. *If* $\mu \in \mathfrak{h}^*$ *is maxl subj to* $V(\mu) \neq 0$*, then* $V(\mu) \subset V^{\mathfrak{n}}$.
4. *If* $V \neq 0, \exists \mu$ *with* $0 \neq V(\mu) \subset V^{\mathfrak{n}}$.
5. $\forall \lambda \in \mathfrak{h}^*$, $\mathrm{Hom}_{\mathfrak{g}}(I_\lambda, V) \simeq \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda, V^{\mathfrak{n}})$.

Parts (1)–(4) guarantee existence of "highest weights;" based on formal calculations with lattices in vector spaces, and $\mathfrak{n} \cdot V(\mu') \subset \sum_{\alpha \in \Delta^+} V(\mu' + \alpha)$.

Sketch of proof of (5):

$\mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, V) \simeq \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, V) = \mathrm{Hom}_{U(\mathfrak{h})}(\mathbb{C}_\lambda, V^{\mathfrak{n}})$.

First isom: "change of rings." Second: $\mathfrak{n} \cdot \mathbb{C}_\lambda =_{\mathrm{def}} 0$.

Moral of the story

For category $\mathcal{O}$, two key ingredients:

1. Highest weight: $V^{\mathfrak{n}} \neq 0$.
2. Universality: $V^{\mathfrak{n}} \rightsquigarrow$ maps from Verma modules.

1st from comb/geom in $\mathfrak{h}^*$, 2nd from homological alg.

Irrs J in $\mathcal{O}$ param by $\lambda \in \mathfrak{h}^*$; characteristic is $\mathbb{C}_\lambda \subset J_\lambda^{\mathfrak{n}}$.

Same two ideas apply to $(\mathfrak{g}, K)$-modules.

Technical problem: change of rings needed is not projective, so $\otimes$ has to be supplemented by Tor.

Parallel problem: construct not $\mathfrak{n}$-fixed vectors, but some derived functors $H^p(\mathfrak{n}, \cdot)$.

Irrs J in $\mathcal{M}(\mathfrak{g}, K)$ param by $\gamma \in \widehat{H}$, some θ-stable Cartan $H \subset G$; characteristic is $\mathbb{C}_\gamma \subset H^s(\mathfrak{n}, J)$.

Lie algebra cohomology

$\mathfrak{n}$ Lie alg (e.g. nil radical of a parabolic in reductive $\mathfrak{g}$.)
Study *functor of $\mathfrak{n}$-invts* $V \mapsto V^{\mathfrak{n}}$ on reps of $\mathfrak{n}$.

Extra structure: $\mathfrak{n} \lhd \mathfrak{b} \implies V^{\mathfrak{n}}$ is $\mathfrak{b}/\mathfrak{n}$-module.

Functor left exact; not right exact unless $\mathfrak{n} = 0$.

Definition 1. $H^p(\mathfrak{n}, \cdot)$ is the pth right derived functor of $\cdot^{\mathfrak{n}}$.

Definition 2. Suppose

$$0 \to V \to I_0 \to \cdots \to I_{p-1} \to I_p \to I_{p+1} \to \cdots$$

is an injective resolution of V as a $U(\mathfrak{n})$-module. Then

$$H^p(\mathfrak{n}, V) = \ker[I_p^{\mathfrak{n}} \to I_{p+1}^{\mathfrak{n}}] / \operatorname{im}[I_{p-1}^{\mathfrak{n}} \to I_p^{\mathfrak{n}}].$$

Definition 3. $H^p(\mathfrak{n}, V) =$ pth coh of cplx $\operatorname{Hom}(\bigwedge^p \mathfrak{n}, V)$.

Extra structure: $\mathfrak{n} \lhd \mathfrak{b} \implies H^p(\mathfrak{n}, V)$ is $\mathfrak{b}/\mathfrak{n}$-module.

$0 \to V_1 \to V_2 \to V_3 \to 0$ exact seq of $\mathfrak{n}$-modules $\implies$

$$\begin{aligned}
0 \longrightarrow H^0(\mathfrak{n}, V_1) &\longrightarrow H^0(\mathfrak{n}, V_2) \longrightarrow H^0(\mathfrak{n}, V_3)\\
\longrightarrow H^1(\mathfrak{n}, V_1) &\longrightarrow H^1(\mathfrak{n}, V_2) \longrightarrow H^1(\mathfrak{n}, V_3)\\
&\vdots\\
\longrightarrow H^d(\mathfrak{n}, V_1) &\longrightarrow H^d(\mathfrak{n}, V_2) \longrightarrow H^d(\mathfrak{n}, V_3) \longrightarrow 0
\end{aligned}$$

Casselman-Osborne theorem

$K \supset G$ max compact in real reductive, θ Cartan invol $\rightsquigarrow$ pair $(\mathfrak{g}, K)$.

$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ Levi decomp of parabolic subalg; assume $\mathfrak{l} = \theta\mathfrak{l} = \bar{\mathfrak{l}}$. Get Levi pair $(\mathfrak{l}, L \cap K)$.

Theorem

Lie algebra cohomology is a cohomological family of functors $H^p(\mathfrak{u}, \cdot) \colon \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{l}, L \cap K)$. Each carries modules of finite length to modules of finite length.

"Finite length" close to "quasisimple." Proof of thm depends on analyzing $\mathfrak{Z}(\mathfrak{g})$...

$U(\mathfrak{g}) = U(\mathfrak{u}) \otimes U(\mathfrak{l}) \otimes U(\mathfrak{u}^-)$ gives linear projection

$$\xi \colon U(\mathfrak{g}) \to U(\mathfrak{l}); \quad \xi \colon U(\mathfrak{g})^{\mathfrak{z}(\mathfrak{l})} \to U(\mathfrak{l})^{\mathfrak{z}(\mathfrak{l})} \text{ alg hom.}$$

Theorem (Casselman-Osborne)

If V is a $\mathfrak{g}$-module, then $\mathfrak{Z}(\mathfrak{g})$ acts on $H^p(\mathfrak{u}, V)$. This action is related to the $\mathfrak{l}$ action by $z \cdot \omega = \xi(z) \cdot \omega$.

Interlude: Chevalley isomorphism

Complex reductive $\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$; $W = W(\mathfrak{g}, \mathfrak{h})$ acts on $\mathfrak{h}, \ \mathfrak{h}^*$.

Example: $\mathfrak{gl}(n) \supset$ upper triang mats $\supset$ diag mats $\simeq \mathbb{C}^n$. $W = S_n$.

$\rho =$ half sum of pos roots $\in \mathfrak{h}^*$. Twisted action $*$ of W is

$$w * \lambda =_{\text{def}} w(\lambda + \rho) - \rho, \quad (w * p)(\lambda) =_{\text{def}} p(w^{-1} * \lambda)$$

$(\lambda \in \mathfrak{h}^*, p \in S(\mathfrak{h}))$.

Example: $\rho = ((n-1)/2, (n-3)/2, \ldots - (n-1)/2)$,
$w * (\lambda_1, \ldots, \lambda_n) = (\ldots, \lambda_{w^{-1}(i)} + (i - w^{-1}(i)), \ldots)$.

Theorem (Chevalley)

The algebra homomorphism $\xi \colon \mathfrak{Z}(\mathfrak{g}) \to S(\mathfrak{h})$ from previous slide is injection with image equal to $S(\mathfrak{h})^{W,*}$, the invts of the twisted W action. Consequently maxl ideals in $\mathfrak{Z}(\mathfrak{g})$ are in one-to-one corr with twisted W orbits on $\mathfrak{h}^*$.

Corollary of Thm and Casselman-Osborne: if $\mathfrak{g}$-module V has infl char $\lambda \in \mathfrak{h}^*$, then $H^p(\mathfrak{u}, V)$ has finite filtration with each level of infl char $w * \lambda$, some $w \in W(\mathfrak{l}, \mathfrak{h}) \backslash W(\mathfrak{g}, \mathfrak{h})$.

Cartan-Weyl and Bott-Kostant

K compact, $\mathfrak{b}_\mathfrak{k} = \mathfrak{t} + \mathfrak{n}_\mathfrak{k}$ Levi decomp of Borel.

Nota bene: automatically $\overline{\mathfrak{n}_\mathfrak{k}} = [\mathfrak{n}_\mathfrak{k}]^-$; defines *complex structure on K/T*, identifying it with projective algebraic complete flag variety $K(\mathbb{C})/(B_K)(\mathbb{C})$.

Write $\Delta^+(\mathfrak{k}, T) = \Delta(\mathfrak{n}_\mathfrak{k}, T)$.

$$X^*(T) = \text{lattice of chars of } T \supset X^*(T)^+_K;$$

$$X^*(T)^+_K =_{\text{def}} \{\mu \in X^*(T) \mid \mu(\alpha^\vee) \geq 0 \ \ (\alpha \in \Delta^+(\mathfrak{k}, T))\}$$

Assume henceforth K connected.

Theorem (Cartan-Weyl)

If K connected, then irr reps of K are param by $X^*(T)^+_K$, by requirement $E_\mu^{\mathfrak{n}_\mathfrak{k}} = \mathbb{C}_\mu$ as rep of T: $H^0(\mathfrak{n}_\mathfrak{k}, E_\mu) = \mathbb{C}_\mu$.

$\leftrightsquigarrow$ Borel-Weil theorem ($E_\mu \subset$hol secs of bdle on K/T).

Theorem (Bott-Kostant)

The only weights of T appearing in $H^*(\mathfrak{n}_\mathfrak{k}, E_\mu)$ are those in $W * \mu$, the twisted W-orbit of the highest weight:

$$\mathbb{C}_{w*\mu} \subset H^p(\mathfrak{n}_\mathfrak{k}, E_\mu) \iff \ell(w) = p.$$

$\leftrightsquigarrow$ Bott theorem ($E_\mu \subset$ Dolbeault coh of bdle on K/T).

Compact gps K: Bott-Kostant continued

K cpt conn, $\mathfrak{q}_\mathfrak{k} = \mathfrak{l}_\mathfrak{k} + \mathfrak{u}_\mathfrak{k}$ Levi decomp of parabolic.

Nota bene: automatically $\overline{\mathfrak{u}_\mathfrak{k}} = [\mathfrak{u}_\mathfrak{k}]^-$; defines *complex structure on $K/L_K \simeq K(\mathbb{C})/(Q_K)(\mathbb{C})$*.

Theorem (Kostant)

If $\mu \in X^*(T)^+_K \leftrightarrow E_\mu$ irr for K, then the only irr reps of L_K appearing in $H^*(\mathfrak{u} \cap \mathfrak{k}, E_\mu)$ are $F_{w*\mu}$, with $w \in W_{L_K} \backslash W_K$ a minimal lgth coset representative. In fact

$$F_{w*\mu} \subset H^p(\mathfrak{u} \cap \mathfrak{k}, E_\mu) \iff \ell(w) = p.$$

Thm equiv to Bott's thm on occurrence of E_μ in Dolbeault cohom of irr holom vec bdles on K/L_K.

First statement of Thm follows from Casselman-Osborne. For second, look at complex for Lie alg cohom:

$F_{w*\mu}$ appears once in $\mathrm{Hom}(\bigwedge \mathfrak{u}_\mathfrak{k}, E_\mu)$, deg $\ell(w)$.

Under the hood: details of Kostant proof

Pf of Kostant thm on Lie alg cohom (either for $\mathfrak{b}$ or for arbitrary $\mathfrak{q}$) was Casselman-Osborne plus

$$F_{w*\mu} \text{ appears once in } \mathrm{Hom}(\textstyle\bigwedge \mathfrak{u}_{\mathfrak{k}}, E_\mu), \text{ deg } \ell(w). \qquad (A)$$

For (A), fix dom reg $r \in \mathfrak{t}$: $\alpha(r) > 0$, all $\alpha \in \Delta^+$. Ask

$$\text{What wts } \gamma \text{ of } \mathrm{Hom}(\textstyle\bigwedge \mathfrak{u}_{\mathfrak{k}}, E_\mu) \text{ maximize } \gamma(wr)? \qquad (C)$$

D, R T-reps $\implies$ wts of $\mathrm{Hom}(D, R)$ are (wts of R) $-$ (wts of D). So break (C) into two questions:

$$\text{what wts } \gamma_E \text{ of } E_\mu \text{ maximize } \gamma_E(wr)? \qquad (CE)$$

$$\text{what wts } \gamma_\wedge \text{ of } \textstyle\bigwedge \mathfrak{u}_{\mathfrak{k}} \text{ minimize } \gamma_\wedge(wr)? \qquad (C\wedge)$$

Answer to (CE) is $w\mu$, mult one.

Define $\Delta^+(w) = \{\alpha \in \Delta^+ \mid \alpha(wr) < 0\}$. Easy to see $|\Delta^+(w)| = \ell(w)$. Since w minimal in $W_{L_K}w$, $\Delta + (w)$ has no roots of $\Delta^+(\mathfrak{l}_{\mathfrak{k}})$. Conclude answer to $(C\wedge)$ is

$$\textstyle\sum_{\alpha \in \Delta^+(w)} \alpha = \rho - w\rho, \qquad \text{(mult one)}.$$

Answer to (C) is $w * \mu$, mult one, deg $\ell(w)$. (A) follows.

Theorem of extremal weights

$G \supset K \supset T_0$ real reductive $\supset$ maxl cpt $\supset$ max torus.

$$H = TA =_{\text{def}} \text{Cent}_G(T_0) \quad \text{fundamental Cartan subgp,}$$
$$W(G,H) \simeq W(K,T) \supset W(K_0,T_0).$$

Fix nondeg real invt bilinear form $\langle , \rangle$ on $\mathfrak{g}_0$, preserved by θ, pos def on $\mathfrak{s}_0$, neg def on $\mathfrak{k}_0$.

Definition

An *extremal wt* of rep E of K is $\mu' \in X^*(T_0)$ with $\langle \mu', \mu' \rangle$ maxl subject to $E(\mu') \neq 0$. If $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{n}_{\mathfrak{k}} + \mathfrak{t}$ is a Borel subalg, a $\mathfrak{b}_{\mathfrak{k}}$*-highest wt* of E is a wt μ of $E^{\mathfrak{n}_{\mathfrak{k}}}$.

Theorem (Cartan-Weyl)

1. *If the extremal wt μ of E is dominant for $\mathfrak{b}_{\mathfrak{k}}$, then $E(\mu) \subset E^{\mathfrak{n}_{\mathfrak{k}}}$; so μ is $\mathfrak{b}_{\mathfrak{k}}$-highest.*
2. *The extremal wts of an irr $E \in \widehat{K}$ form one $W(K,T)$ orbit. These weights have multiplicity one.*
3. *Extremal wts make finite-to-one correspondence $\widehat{K} \twoheadrightarrow X^*(T_0)/W(K,T)$.*

Top cohomology

$K \supset T_0$ maxl torus, $s = (\dim K/T)/2$.

Definition

A *top cohomology weight* for rep E of K is a weight of max lgth in $H^*(\mathfrak{n}_\mathfrak{k}, E)$, with $\mathfrak{b}_\mathfrak{k} \supset \mathfrak{t}$ Borel subalgebra. If E has to cohom wt γ, the *top cohomology norm* for E is $\|E\|_{\text{top}} =_{\text{def}} \langle \gamma, \gamma \rangle$.

Proposition

Suppose $E \in \widehat{K}$ has extr wt μ. The top cohomology weights of E are those in $W \cdot (\mu + 2\rho_c)$, with $2\rho_c$ sum of a set of pos roots making μ dominant. Precisely, $\mu + 2\rho_c$ appears in $H^p(\mathfrak{n}_\mathfrak{k}, E) \iff 2\rho_c \leftrightsquigarrow \mathfrak{b}_\mathfrak{k}^-$ and $p = s$. In particular, $\|E\|_{top} =_{def} \langle \mu + 2\rho_c, \mu + 2\rho_c \rangle$.

Notice $\mathfrak{b}_\mathfrak{k}^-$: to get top degree cohomology, largest possible weight, must use Borel *opposite* to one making μ dominant.

It is these cohomology classes, not highest weights that will be generalized to $(\mathfrak{g}, K)$ modules.

Definition of lowest K-type

Want idea like "extremal weights" for $(\mathfrak{g}, K)$-mods.

$G \supset K \supset T_0$, $H = TA =_{\text{def}} \text{Cent}_G(T_0)$ fundamental Cartan.

Definition

If V is a $(\mathfrak{g}, K)$-module, *lowest K-type* is $E \in \widehat{K}$ that has $\|E\|_{\text{top}}$ minimal subject to the reqt that E appear in V.

Set of values of a pos def quad form on a lattice is discrete and nonnegative. It follows that *every nonzero $V \in \mathcal{M}(\mathfrak{g}, K)$ has at least one lowest K-type*.

Fix $\mu \in X^*$, $\Delta^+(\mathfrak{k}, T_0)$ making μ dom, $\rightsquigarrow 2\rho_c =$ sum of pos roots. Choose $\Delta^+(\mathfrak{g}, T)$ making $\mu + 2\rho_c$ dominant $\rightsquigarrow \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ θ-stable Borel subalgebra. Prev slide $\Longrightarrow$

$$\dim H^s(\mathfrak{n}_{\mathfrak{k}}^-, V)(\mu + 2\rho_c) = \text{sum of mults of } K \text{ reps of top cohom wt } \mu + 2\rho_c$$

These $\mathfrak{n}_{\mathfrak{k}}^-$ cohom classes perfectly identify some (lowest) K-types of V. But $\mathfrak{n}_{\mathfrak{k}}^-$ *too small* to identify V in this way.

Cohomology and lowest K-types: generic μ

Continue with $V \in \mathcal{M}(\mathfrak{g}, K)$, lowest K-type E of top cohom wt $\mu + 2\rho_c \rightsquigarrow \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ θ-stable Borel subalgebra. Write $\rho \in \mathfrak{t}^*$ for half sum of roots in $\mathfrak{n}^-$.

Theorem

If in addition $\mu + 2\rho_c - \rho$ has str pos inner prod with each positive root, then the natural restriction map

$$\text{res}\colon H^s(\mathfrak{n}^-, V) \to H^s(\mathfrak{n}_{\mathfrak{k}}^-, V)$$

is injective on $\mu + 2\rho_c$ wt space of T_0.

Proof follows proof of Kostant theorem.

In place of standard complex to compute $H^\cdot(\mathfrak{n}^-, V)$, use Hochschild-Serre spectral sequence: E^1 term is

$$H^p(\mathfrak{n}_{\mathfrak{k}}^-, V) \otimes \bigwedge^q(\mathfrak{n}_{\mathfrak{s}}^-) \quad (\mathfrak{g} = \mathfrak{k} + \mathfrak{s} \text{ Cartan decomp}).$$

Spectral sequence is T-eqvt. Hypothesis E lowest, plus Kostant desc of $H^p(\mathfrak{n}_{\mathfrak{k}}^-, V) \implies$ wt $\mu + 2\rho_c$ appears in E^1 only in degree $(s, 0)$. Thm follows.

Langlands classif: $V \rightsquigarrow$ char of H on $H^s(\mathfrak{n}^-, V)$.

Cohomology and lowest K-types: all μ

Continue with $V \in \mathcal{M}(\mathfrak{g}, K)$, lowest K-type E of top cohom wt $\mu + 2\rho_c \rightsquigarrow \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ θ-stable Borel subalgebra. Write $\rho \in \mathfrak{t}^*$ for half sum of roots in $\mathfrak{n}^-$.

$\lambda =$ orth proj of $\mu + 2\rho_c - \rho$ on pos Weyl chamber.

"Generic case" was $\lambda = \mu + 2\rho_c - \rho$ in interior of chamber.

Define $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \supset \mathfrak{b} =$ parabolic subalg defined by λ.

(Re)define $s = \dim \mathfrak{u}_{\mathfrak{k}}$, $F =$ irr of L_K of top cohom wt $\mu + 2\rho_c$ appearing in $H^s(\mathfrak{u}_{\mathfrak{k}}, E)$.

$$\dim\left(\mathrm{Hom}_{L_K}(F, H^s(\mathfrak{u}_{\mathfrak{k}}^-, V))\right) = \text{sum of mults of } K \text{ reps of top cohom wt } F.$$

Theorem

res: $H^s(\mathfrak{u}^-, V) \to H^s(\mathfrak{u}_{\mathfrak{k}}^-, V)$ *is inj on L_K-isotypic space for F.*

Lowest cohom class of $V =_{\text{def}}$ irr $(\mathfrak{l}, L_K)$ module V_L in $H^s(\mathfrak{u}^-, V)$, containing L_K-type F.

Langlands param for $V =_{\text{def}}$ Langlands param for L; turns out (missing slides!) to be char of max split Cartan $H_L \subset L$.

Why cohomology can identify a module

$K \supset G$ max cpt in real red, θ Cartan inv $\rightsquigarrow$ pair $(\mathfrak{g}, K)$;.

FIX $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ Levi decomp of θ-stable parabolic subalg.
Get $\mathfrak{q}_{\mathfrak{k}} = \mathfrak{l}_{\mathfrak{k}} + \mathfrak{u}_{\mathfrak{k}}$ parabolic in $\mathfrak{k}$, Levi pair $(\mathfrak{l}, L_K)$; $s = \dim \mathfrak{u}_{\mathfrak{k}}$.

FIX irr $(\mathfrak{g}, K)$-module V, (τ, E_τ) irr for $(\mathfrak{k}, L_K)$,
$V_\tau = \mathrm{Hom}_{\mathfrak{k}, L_K}(E_\tau, V)$ multiplicity space.

RECALL CENTRALIZER ALGS:

if $\neq 0$, $V_\tau =$ irr $R(\mathfrak{g}, L_K)^{\mathfrak{k}, L_K}$ module, determines V. $\quad (CENT)$

Theorem

Suppose $X \in \mathcal{M}(\mathfrak{g}, L_K)$; fix coh class $\omega \in H^r(\mathfrak{u}, X) \in \mathcal{M}(\mathfrak{l}, L_K)$.

1. *$H^r(\mathfrak{u}, X)$ is $(\mathfrak{l}, L_K)$-module, so $R(\mathfrak{l}, L_K)$ acts.*
2. *$H^r(\mathfrak{u}_{\mathfrak{k}}, X)$ is $(\mathfrak{l}, L_K)$-module, so $R(\mathfrak{l}_{\mathfrak{k}}, L_K)$ acts.*
3. *$\exists$ restriction homomorphism $\mathrm{res}\colon H^r(\mathfrak{u}, X) \to H^r(\mathfrak{u}_{\mathfrak{k}}, X)$, respecting L_K action.*
4. *$\exists$ Chevalley homomorphism $\xi\colon R(\mathfrak{g}, L_K)^{\mathfrak{k}, L_K} \to R(\mathfrak{l}, L_K)^{L_K}$.*
5. *$R(\mathfrak{g}, L_K)^{\mathfrak{k}, L_K}$ acts on $H^r(\mathfrak{u}_{\mathfrak{k}}, X)$, on range of rep in $\mathrm{Hom}_{L_K}(\bigwedge^r \mathfrak{u}_{\mathfrak{k}}, X)$; commutes with L_K.*
6. *$T \cdot \mathrm{res}(\omega) = \mathrm{res}(\xi(T) \cdot \omega)$.*

If $\mathrm{res}(\omega) \in$ (top) cohom for K rep E_τ, LEFT is action in $(CENT)$. RIGHT is corr cent action for L on cohom.

Forms and dual spaces

V cplx vec space (or alg rep of K, or $(\mathfrak{g}, K)$-module).

Hermitian dual of V

$$V^h = \{\xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v)\}$$

(V alg K-rep $\rightsquigarrow$ require ξ K-finite; V topolog. $\rightsquigarrow$ require ξ cont.)

Sesquilinear pairings between V and W

$$\text{Sesq}(V, W) = \{\langle , \rangle : V \times W \to \mathbb{C}, \text{lin in } V, \text{conj-lin in } W\}$$

$$\text{Sesq}(V, W) \simeq \text{Hom}(V, W^h), \quad \langle v, w \rangle_T = (Tv)(w).$$

Cplx conj of forms is (conj linear) isom

$$\text{Sesq}(V, W) \simeq \text{Sesq}(W, V).$$

Corresponding (conj lin) isom is Hermitian transpose:

$$\text{Hom}(V, W^h) \simeq \text{Hom}(W, V^h), \quad (T^h w)(v) = \overline{(Tv)(w)}.$$

$$(TS)^h = S^h T^h, \qquad (zT)^h = \overline{z}(T^h).$$

Sesq form $\langle , \rangle_T$ on V ($\leftrightsquigarrow$ $T \in \text{Hom}(V, V^h)$) Hermitian if

$$\langle v, v' \rangle_T = \overline{\langle v', v \rangle_T} \iff T^h = T.$$

Defining Herm dual repn(s)

(π, V) $(\mathfrak{g}, K)$-module; Recall Herm dual V^h of V.

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^h, V^h)$$

using Hermitian transpose map of operators.

Definition REQUIRES twisting by conj lin antiaut of $\mathfrak{g}$, gp antiaut of K.

Since $\mathfrak{g}$ equipped with a real form $\mathfrak{g}_0$, have natural conj-lin aut $\sigma_0(X + iY) = X - iY$ $(X, Y \in \mathfrak{g}_0)$. Also $X \mapsto -X$ is Lie alg antiaut, and $k \mapsto k^{-1}$ gp antiaut.

Define $(\mathfrak{g}, K)$-module π^h on V^h,

$$\pi^h(Z) \cdot \xi =_{\text{def}} [\pi(-\sigma_0(Z))]^h \cdot \xi \quad (Z \in \mathfrak{g},\ \xi \in V^h),$$

$$\pi^h(k) \cdot \xi =_{\text{def}} [\pi(k^{-1})]^h \cdot \xi \quad (k \in K,\ \xi \in V^h).$$

Will need also a variant: suppose τ involutive aut of G preserving K. Define $(\mathfrak{g}, K)$-module $\pi^{h,\tau}$ on V^h,

$$\pi^{h,\tau}(X) \cdot \xi = [\pi(-\tau(\sigma_0(Z))]^h \cdot \xi \quad (Z \in \mathfrak{g},\ \xi \in V^h),$$

$$\pi^{h,\tau}(k) \cdot \xi = [\pi(\tau(k)^{-1})]^h \cdot \xi \quad (k \in K,\ \xi \in V^h).$$

Invariant Hermitian forms

$V = (\mathfrak{g}, K)$-module, τ involutive aut of $(\mathfrak{g}, K)$.

A τ-invt sesq form on V is sesq pairing $\langle , \rangle^\tau$ such that

$$\langle Z \cdot v, w \rangle = \langle v, -\tau(\sigma_0(Z)) \cdot w \rangle, \quad \langle k \cdot v, w \rangle = \langle v, \tau(k^{-1}) \cdot w \rangle$$

$$(Z \in \mathfrak{g}; k \in K; v, w \in V).$$

Proposition

τ-invt sesq form on V $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-map $T\colon V \to V^{h,\tau}$:

$$\langle v, w \rangle_T = (Tv)(w).$$

Form is Hermitian $\iff$ $T^h = T$.

Assume from now on V is irreducible.

$V \simeq V^{h,\tau}$ $\iff$ $\exists\tau$-invt sesq $\iff$ $\exists\tau$-invt Herm

τ-invt Herm form on V unique up to real scalar mult.

$T \to T^h \leftrightsquigarrow$ real form of cplx line $\mathrm{Hom}_{\mathfrak{g},K}(V, V^{h,\tau})$.

Deciding existence of τ-invt Hermitian form amounts to computing the involution $V \mapsto V^{h,\tau}$ on $\widehat{G}$.

Hermitian forms and unitary reps

π rep of G on complete loc cvx V_π, V_π^h Hermitian dual space. Hermitian dual reps are (τ inv aut of (G, K))

$$\pi^h(g) = \pi(g^{-1})^h, \qquad \pi^{h,\tau}(g) = \pi(\tau(g^{-1})^h$$

Definition

A *τ-invariant form* is continuous Hermitian pairing

$$\langle , \rangle_\pi^\tau \colon V_\pi \times V_\pi \to \mathbb{C}, \quad \langle \pi(g)v, w\rangle_\pi^\tau = \langle v, \pi(\tau(g^{-1}))w\rangle_\pi^\tau.$$

Equivalently: $T \in \mathrm{Hom}_G(V_\pi, V_\pi^{h,\tau})$, $T = T^h$.

Because infl equiv easier than topol equiv, $V_\pi \simeq V_\pi^{h,\tau} \not\Rightarrow$ existence of a continuous map $V_\pi \to V_\pi^h$. So invt forms may not exist on topological reps even if they exist on $(\mathfrak{g}, K)$-modules.

Theorem (Harish-Chandra)

Passage to K-finite vectors defines bijection from the unitary dual $\widehat{G}_u$ onto equivalence classes of irreducible $(\mathfrak{g}, K)$ modules admitting a pos def invt Hermitian form.

Despite warning, get perfect alg param of $\widehat{G}_u$.

Hermitian duals for $SL(2,\mathbb{R})$

Recall ρ^ν ($\nu \in \mathbb{C}$) family of reps of $SL(2,\mathbb{R})$ defined on

$W = \text{even trig polys on } S^1 = \text{span}(w_m(\theta) = e^{im\theta}, m \in 2\mathbb{Z})$

Rotation by θ in $SO(2)$ acts on w_m by $e^{im\theta}$, Lie alg acts by

$$\rho^\nu(H)w_m = mw_m, \qquad \rho^{\nu,h}(H)w_m = mw_m,$$

$$\rho^\nu(X)w_m = \frac{1}{2}(m+\nu)w_{m+2}, \qquad \rho^{\nu,h}(X)w_m = \frac{1}{2}(m+2-\overline{\nu})w_{m+2},$$

$$\rho^\nu(Y)w_m = \frac{1}{2}(-m+\nu)w_{m-2} \qquad \rho^{\nu,h}(Y)w_m = \frac{1}{2}(-m+2-\overline{\nu})w_{m-2}.$$

Can identify $W \simeq W^h$ by obvious pos def inner product

$$\langle \sum_r a_r w_r, \sum_s b_s w_s \rangle = \sum_p a_p \overline{b_p}.$$

Herm trans: $T = (t_{ij}) \rightsquigarrow T^h = {}^t\overline{T} = (\overline{t_{ji}})$; Herm dual rep ↑.

See that $(\rho^\nu)^h = \rho^{2-\overline{\nu}}$. So $\nu - 1$ imag $\implies$ ρ^ν Herm; in fact form is pos def, so ρ^ν unitary ($\nu \in 1 + i\mathbb{R}$). These are unitary principal series.

There is more to say!

$\nu - 1$ non-imaginary

Calculated $(\rho^\nu)^h = \rho^{2-\overline{\nu}}$. Saw that $\nu \in 1 + i\mathbb{R}$ gave unitary principal series. But we know that ρ^ν is approximately isomorphic to $\rho^{-\nu+2}$; so expect to find more Hermitian representations for $\nu \in \mathbb{R}$.

Theorem (Knapp-Stein)

$\exists!$ *merom fam of lin maps char by*

$$A(\nu)\colon W \to W, \quad A(\nu)\rho^\nu = \rho^{2-\nu} A(\nu), \quad A(\nu) w_0 = w_0.$$

$A(\nu)$ has simple zero spanned by submodule $\{w_m \mid |m| \geq m_0\}$, ($\nu = m_0 = 2, 4, 6 \ldots$)
$A(\nu)$ is finite only on submodule spanned by $\{w_m \mid |m| < m_0\}$ ($\nu = -m_0 + 2 = 0, -2, -4, \ldots 2$); simple pole on quotient.

Form $\langle , \rangle^\nu$ for ρ^ν is std form on W, *twisted by $A(\nu)$*:

$$\langle w_1, w_2 \rangle^\nu = \langle A(\nu) w_1, w_2 \rangle \qquad (\nu \in \mathbb{R}).$$

Question of whether $\langle , \rangle^\nu$ is positive, and the signature, changes with ν (zeros, poles of $A(\nu)$).

Signatures for $SL(2, \mathbb{R})$

Recall $E(\nu) = (\nu^2 - 1)$-eigenspace of $\Delta_{\mathbb{H}}$.

Need "signature" of Herm form on this inf-diml space.

Harish-Chandra (or Fourier) idea:
use $K = SO(2)$ break into fin-diml subspaces

$$E(\nu)_{2m} = \{f \in E(\nu) \mid \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot f = e^{2im\theta} f\}.$$

$$E(\nu) \supset \sum_m E(\nu)_m, \qquad \text{(dense subspace)}$$

Decomp is orthogonal for any invariant Herm form.

Signature is $+$ or $-$ for each m. For $3 < |\nu| < 5$

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$
$\cdots$	$+$	$+$	$-$	$+$	$-$	$+$	$+$	$\cdots$

Deforming signatures for $SL(2, \mathbb{R})$

Here's how signatures of the reps $E(\nu)$ change with ν.

$\nu = 0$, $E(0)$ "$\subset$" $L^2(\mathbb{H})$: unitary, signature positive.

$0 < \nu < 1$, $E(\nu)$ irr: signature remains positive.

$\nu = 1$, form finite pos on $J(1)$ $\leftrightsquigarrow$ $SO(2)$ rep 0.

$\nu = 1$, form has pole, pos residue on $E(1)/J(1)$.

$1 < \nu < 3$, across pole at $\nu = 1$, signature changes.

$\nu = 3$, Herm form finite $- + -$ on $J(3)$.

$\nu = 3$, Herm form has pole, neg residue on $E(3)/J(3)$.

$3 < \nu < 5$, across pole at $\nu = 3$, signature changes. ETC.

Conclude: $J(\nu)$ unitary, $\nu \in [0, 1]$; nonunitary, $\nu \in [1, \infty)$.

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$	$SO(2)$ reps
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$\nu = 0$
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$0 < \nu < 1$
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$\nu = 1$
$\cdots$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$\cdots$	$1 < \nu < 3$

From $SL(2,\mathbb{R})$ to reductive G

Calculated signatures of invt Herm forms on spherical reps of $SL(2,\mathbb{R})$.

Seek to do "same" for real reductive group. Need. . .

List of irr reps $=$ ctble union (cplx vec space)/(fin grp).

reps for purely imag points "$\subset$" $L^2(G)$: unitary!

Natural (orth) decomp of any irr (Herm) rep into fin-diml subspaces $\rightsquigarrow$ define signature subspace-by-subspace.

Signature at $\nu + i\tau$ by analytic cont $t\nu + i\tau$, $0 \le t \le 1$.

Precisely: start w unitary (pos def) signature at $t = 0$; add contribs of sign changes from zeros/poles of odd order in $0 \le t \le 1 \rightsquigarrow$ signature at $t = 1$.

Character formulas

Can decompose Verma module into irreducibles

$$I_\lambda = \sum_{\mu \le \lambda} m_{\mu,\lambda} J_\mu \qquad (m_{\mu,\lambda} \in \mathbb{N})$$

or write a formal character for an irreducible

$$J_\lambda = \sum_{\mu \le \lambda} M_{\mu,\lambda} I_\mu \qquad (M_{\mu,\lambda} \in \mathbb{Z})$$

Can decompose standard HC module into irreducibles

$$I(x) = \sum_{y \le x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N}).$$

Here x and y are params for irreducible $(\mathfrak{g}, K)$-mods, or (what is the same thing!) params for std $(\mathfrak{g}, K)$-mods.

Similarly, can write a formal character for an irreducible

$$J(x) = \sum_{y \le x} M_{y,x} I(y) \qquad (M_{y,x} \in \mathbb{Z}).$$

Matrices m and M upper triang, ones on diag, mutual inverses. Entries are KL polynomials eval at 1:

$$m_{y,x} = Q_{y,x}(1), \quad M_{y,x} = \pm P_{y,x}(1) \quad (Q_{y,x}, P_{y,x} \in \mathbb{N}[q]).$$

Last statements most literally true at reg infl char, but Jantzen/Zuckerman transl princ gives sing. infl char.

Character formulas for $SL(2,\mathbb{R})$

Std $(\mathfrak{g},K)$-mods include princ series

$$W^{\nu-1} =_{\text{def}} I_{\text{ev}}(\nu) \twoheadrightarrow J(\nu) \qquad (\text{Re}(\nu) \geq 0);$$

Langlands quotient $J(\nu) = I(\nu)$ except for $\nu = 2m+1\ldots$, when $J(\nu)$ has dim $2m+1$.
Need discrete series $I_\pm(n)$ ($n = 1, 2, \ldots$) char by

$$I_+(n)|_{SO(2)} = n+1,\ n+3,\ n+5\cdots$$
$$I_-(n)|_{SO(2)} = -n-1,\ -n-3,\ -n-5\cdots$$

Discrete series reps are irr: $I_\pm(n) = J_\pm(n)$
Decompose principal series

$$I_{\text{ev}}(2m+1) = J_{\text{ev}}(2m+1) + J_+(2m+1) + J_-(2m+1).$$

Character formula

$$J_{\text{ev}}(2m+1) = I_{\text{ev}}(2m+1) - I_+(2m+1) - I_-(2m+1).$$

$\pm P_{x,y}$	$I_{\text{ev}}(2m+1)$	$I_+(2m+1)$	$I_-(2m+1)$
$J_{\text{ev}}(2m+1)$	1	-1	-1
$J_+(2m+1)$	0	1	0
$J_-(2m+1)$	0	0	1

Invariant forms on standard reps

Recall multiplicity formula

$$I(x) = \sum_{y \leq x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N})$$

for standard $(\mathfrak{g}, K)$-mod $I(x)$.

Want parallel formulas for σ-invt Hermitian forms.

Need forms on standard modules.

Form on irr $J(x) \xrightarrow{\text{deformation}}$ Jantzen filt $I^k(x)$ on std, nondeg forms $\langle, \rangle^k$ on I^k/I^{k+1}.

Details (proved by Beilinson-Bernstein):

$$I(x) = I^0 \supset I^1 \supset I^2 \supset \cdots, \qquad I^0/I^1 = J(x)$$

$$I^k/I^{k+1} \text{ completely reducible}$$

$$[J(y) \colon I^k/I^{k+1}] = \text{coeff of } q^{(\ell(x)-\ell(y)-k)/2} \text{ in KL poly } Q_{y,x}$$

Hence $\langle, \rangle_{I(x)} \overset{\text{def}}{=} \sum_k \langle, \rangle^k$, nondeg form on gr $I(x)$.

Restricts to original form on irr $J(x)$.

Virtual Hermitian forms

$$\mathbb{Z} = \text{Groth group of vec spaces.}$$

These are mults of irr reps in virtual reps.

$$\mathbb{Z}[X] = \text{Groth grp of finite length reps.}$$

For invariant forms...

$$\mathbb{W} = \mathbb{Z} \oplus \mathbb{Z} = \begin{array}{c}\text{Grothendieck group of}\\ \text{finite-dimensional forms.}\end{array}$$

Ring structure

$$(p, q)(p', q') = (pp' + qq', pq' + q'p).$$

Mult of irr-with-forms in virtual-with-forms is in $\mathbb{W}$:

$$\mathbb{W}[X] \approx \text{Groth grp of fin lgth reps with invt forms.}$$

Two problems: invt form $\langle , \rangle_J$ may not exist for irr J; and $\langle , \rangle_J$ may not be preferable to $-\langle , \rangle_J$.

What's a Jantzen filtration?

V cplx, $\langle , \rangle_t$ Herm forms analytic in t, generically nondeg.

$$V = V^0(t) \supset V^1(t) = \mathrm{Rad}(\langle , \rangle_t), \quad J(t) = V^0(t)/V^1(t)$$

$$(p^0(t), q^0(t)) = \text{signature of } \langle , \rangle_t \text{ on } J(t).$$

Question: how does $(p^0(t), q^0(t))$ change with t?

First answer: locally constant on open set $V^1(t) = 0$.

Refine answer...define form on $V^1(t)$

$$\langle v, w \rangle^1(t) = \lim_{s \to t} \frac{1}{t-s} < v, w >_s, \qquad V_2(t) = \mathrm{Rad}(\langle , \rangle^1(t))$$

$$(p^1(t), q^1(t)) = \text{signature of } \langle , \rangle^1(t).$$

Continuing gives Jantzen filtration

$$V = V^0(t) \supset V^1(t) \supset V^2(t) \cdots \supset V^{m+1}(t) = 0$$

From $t - \epsilon$ to $t + \epsilon$, signature changes on odd levels:

$$p(t+\epsilon) = p(t-\epsilon) + \sum [-p^{2k+1}(t) + q^{2k+1}(t)].$$

Hermitian KL polynomials: multiplicities

Fix invt Hermitian form $\langle , \rangle_{J(x)}$ on each irr having one; recall Jantzen form $\langle , \rangle^n$ on $I(x)^n / I(x)^{n+1}$.

MODULO problem of irrs with no invt form, write

$$(I^n / I^{n+1}, \langle , \rangle^n) = \sum_{y \le x} w_{y,x}(n)(J(y), \langle , \rangle_{J(y)}),$$

coeffs $w(n) = (p(n), q(n)) \in \mathbb{W}$; summand means

$$p(n)(J(y), \langle , \rangle_{J(y)}) \oplus q(n)(J(y), -\langle , \rangle_{J(y)})$$

Define Hermitian KL polynomials

$$Q^h_{y,x} = \sum_n w_{y,x}(n) q^{(l(x)-l(y)-n)/2} \in \mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q = 1 \leftrightarrow$ form $\langle , \rangle_{I(x)}$ on std.

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W} \to \mathbb{Z} \leftrightarrow$ KL poly $Q_{y,x}$.

Hermitian KL polynomials: characters

Matrix $Q^h_{y,x}$ is upper tri, 1s on diag: INVERTIBLE.

$$P^h_{x,y} \stackrel{\text{def}}{=} (-1)^{l(x)-l(y)}((x,y) \text{ entry of inverse}) \in \mathbb{W}[q].$$

Definition of $Q^h_{x,y}$ says

$$(\text{gr}\, I(x), \langle,\rangle_{I(x)}) = \sum_{y \le x} Q^h_{x,y}(1)(J(y), \langle,\rangle_{J(y)});$$

inverting this gives

$$(J(x), \langle,\rangle) = \sum_{y \le x} (-1)^{l(x)-l(y)} P^h_{x,y}(1)(\text{gr}\, I(y), \langle,\rangle).$$

Next question: how do you compute $P^h_{x,y}$?

Herm KL polys for σ_c

$\sigma_c =$ cplx conj for cpt form of G, $\sigma_c(K) = K$.

Plan: study σ_c-invt forms, relate to σ_0-invt forms.

Proposition

Suppose $J(x)$ irr $(\mathfrak{g}, K)$-module, real infl char. Then $J(x)$ has σ_c-invt Herm form $\langle , \rangle^c_{J(x)}$, characterized by

$\langle , \rangle^c_{J(x)}$ *is pos def on the lowest K-types of $J(x)$.*

Proposition $\implies$ Herm KL polys $Q^c_{x,y}$, $P^c_{x,y}$ well-def.

Coeffs in $\mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z}$; $s = (0,1) \leftrightsquigarrow$ one-diml neg def form.

Conj: $Q^c_{x,y}(q) = s^{\frac{\ell_o(x)-\ell_o(y)}{2}} Q_{x,y}(qs), \quad P^c_{x,y}(q) = s^{\frac{\ell_o(x)-\ell_o(y)}{2}} P_{x,y}(qs).$

Equiv: if $J(y)$ occurs at level k of Jantzen filt of $I(x)$, then Jantzen form is $(-1)^{(l(x)-l(y)-k)/2}$ times $\langle , \rangle_{J(y)}$.

Conjecture is false... but not seriously so. Need an extra power of s on the right side.

Orientation number

Conjecture $\leftrightarrow$ KL polys $\leftrightarrow$ *integral* roots.

Simple form of Conjecture $\Rightarrow$ Jantzen-Zuckerman translation across non-integral root walls preserves signatures of (σ_c-invariant) Hermitian forms.

It ain't necessarily so.

$SL(2,\mathbb{R})$: translating spherical principal series from (real non-integral positive) ν to (negative) $\nu - 2m$ changes sign of form iff $\nu \in (0,1) + 2\mathbb{Z}$.

Orientation number $\ell_o(x)$ is

1. # pairs $(\alpha, -\theta(\alpha))$ cplx nonint, pos on x; PLUS
2. # real β s.t. $\langle x, \beta^\vee\rangle \in (0,1) + \epsilon(\beta, x) + 2\mathbb{N}$.

$\epsilon(\beta, x) = 0$ spherical, 1 non-spherical.

Deforming to $\nu = 0$

Have computable conjectural formula (omitting ℓ_o)

$$(J(x), \langle,\rangle^c_{J(x)}) = \sum_{y \le x} (-1)^{l(x)-l(y)} P^c_{x,y}(s)(\mathrm{gr}\, I(y), \langle,\rangle^c_{I(y)})$$

for σ^c-invt forms in terms of forms on stds, same inf char.

Polys $P^c_{x,y}$ are KL polys, computed by `atlas` software.

Std rep $I = I(\nu)$ deps on cont param ν. Put $I(t) = I(t\nu),\ t \ge 0$.

Apply Jantzen formalism to deform t to 0...

$$\langle,\rangle^c_J = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle,\rangle^c_{I'(0)} \qquad (v_{J,I'} \in \mathbb{W}).$$

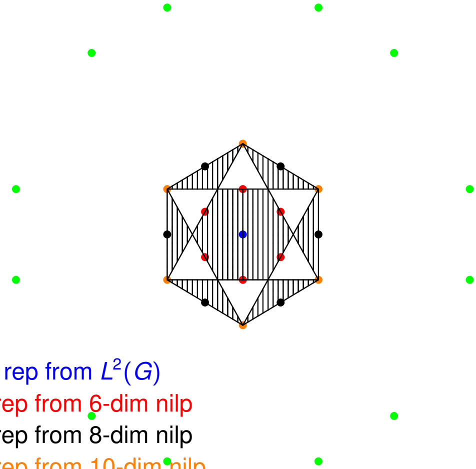
More rep theory gives formula for $G(\mathbb{R})$-invt forms:

$$\langle,\rangle^0_J = \sum_{I'(0) \text{ std at } \nu' = 0} s^{\epsilon(I')} v_{J,I'} \langle,\rangle^0_{I'(0)}.$$

$I'(0)$ unitary, so J unitary $\iff$ all coeffs are $(p, 0) \in \mathbb{W}$.

Example of $G_2(\mathbb{R})$

Real parameters for spherical unitary reps of $G_2(\mathbb{R})$



- Unitary rep from $L^2(G)$
- Arthur rep from 6-dim nilp
- Arthur rep from 8-dim nilp
- Arthur rep from 10-dim nilp
- Trivial rep

From σ_c to σ_0

Cplx conjs σ_c (compact form) and σ_0 (our real form) differ by Cartan involution θ: $\sigma_0 = \theta \circ \sigma_c$.

Irr $(\mathfrak{g}, K)$-mod $J \rightsquigarrow J^\theta$ (same space, rep twisted by θ).

Proposition

J admits σ_0-invt Herm form if and only if $J^\theta \simeq J$. If $T_0 \colon J \xrightarrow{\sim} J^\theta$, and $T_0^2 = \mathrm{Id}$, then

$$\langle v, w \rangle_J^0 = \langle v, T_0 w \rangle_J^c.$$

$T \colon J \xrightarrow{\sim} J^\theta \Rightarrow T^2 = z \in \mathbb{C} \Rightarrow T_0 = z^{-1/2} T \rightsquigarrow \sigma$-invt Herm form.

To convert formulas for σ_c invt forms $\rightsquigarrow$ formulas for σ_0-invt forms need intertwining ops $T_J \colon J \xrightarrow{\sim} J^\theta$, consistent with decomp of std reps.

Equal rank case

rk K = rk $G \Rightarrow$ Cartan inv inner: $\exists \tau \in K$, $\mathrm{Ad}(\tau) = \theta$.

$\theta^2 = 1 \Rightarrow \tau^2 = \zeta \in Z(G) \cap K$.

Study reps π with $\pi(\zeta) = z$. Fix square root $z^{1/2}$.

If ζ acts by z on V, and $\langle , \rangle_V^c$ is σ_c-invt form, then $\langle v, w\rangle_V^0 \overset{\text{def}}{=} \langle v, z^{-1/2}\tau \cdot w\rangle_V^c$ is σ_0-invt form.

$$\langle , \rangle_J^c = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle_{I'(0)}^c \qquad (v_{J,I'} \in \mathbb{W}).$$

translates to

$$\langle , \rangle_J^0 = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle_{I'(0)}^0 \qquad (v_{J,I'} \in \mathbb{W}).$$

I' has LKT $\mu' \Rightarrow \langle , \rangle_{I'(0)}^0$ definite, sign $z^{-1/2}\mu'(\tau)$.

J unitary $\iff$ each summand on right pos def.

Computability of $v_{J,I'}$ needs conjecture about $P_{x,y}^{\sigma_c}$.

General case

Fix "distinguished involution" δ_0 of G inner to θ

Define extended group $G^\Gamma = G \rtimes \{1, \delta_0\}$.

Can arrange $\theta = \mathrm{Ad}(\tau\delta_0)$, some $\tau \in K$.

Define $K^\Gamma = \mathrm{Cent}_{G^\Gamma}(\tau\delta_0) = K \rtimes \{1, \delta_0\}$.

Study $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-mods V with $D_0 \colon V \xrightarrow{\sim} V^{\delta_0}$, $D_0^2 = \mathrm{Id}$.

Beilinson-Bernstein localization: $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ action of δ_0 on K-eqvt perverse sheaves on G/B.

Should be computable by mild extension of Kazhdan-Lusztig ideas. Not done yet!

Now translate σ_c-invt forms to σ_0 invt forms

$$\langle v, w\rangle_V^0 \overset{\mathrm{def}}{=} \langle v, z^{-1/2}\tau\delta_0 \cdot w\rangle_V^c$$

on $(\mathfrak{g}, K^\Gamma)$-mods as in equal rank case.

Kazhdan-Lusztig polys and Bruhat order

Classical KL polynomials are $P_{y,w}$, with y and w in a Weyl group W. They satisfy

$$P_{y,w} \neq 0 \iff y \leq w, \qquad P_{y,w}(0) = \begin{cases} 1 & y \leq w \\ 0 & y \not\leq w \end{cases}.$$

Here $\leq$ is the Bruhat order. The statements about value at 0 are related to the Möbius function for the Bruhat order.

KL polynomials for a real reductive G are $P_{\gamma',\gamma}$, with γ' and γ in a block $\mathbb{B}$ of irreducible $(\mathfrak{g}, K)$ modules of regular infinitesimal character. This block has a Bruhat order. The polynomials satisfy

1. Bruhat ord is the transitive closure of rel $P_{\gamma',\gamma} \neq 0$.
2. Conjecturally $P_{\gamma',\gamma}(0)$ is zero or a power of 2.

prove the conjecture, and understand what KL polynomials have to do with the Möbius function for $\mathcal{B}$.

Twisted Kazhdan-Lusztig polynomials

Fix involutive automorphism δ of (W, S), and write $\mathcal{I}_\delta$ for the set of twisted involutions in W.

Lusztig and I recently introduced variants $P^\delta_{y,w}$ of KL polynomials, indexed by $y, w \in \mathcal{I}_\delta$.

Real reductive $G \rightsquigarrow$ involutive aut δ of (W, S) (action of the Cartan inv on a θ-stable Borel subalgebra).

For a block $\mathbb{B}$ at integral infl char, $\exists \tau \colon \mathbb{B} \to \mathcal{I}_\delta$; τ is surjective if (and only if) G is quasisplit and $\mathbb{B}$ includes fundamental series representations.

Relate $P_{\gamma',\gamma}$ to $P^\delta_{\tau(\gamma'),\tau(\gamma)}$.

Computing less

atlas software computes KL polynomials $P_{\gamma',\gamma}$ for all $\gamma', \gamma \in \mathcal{B}$.

This is a LOT of information (tens of gigabytes for split real E_8); hard to know what is interesting.

One thing that's interesting: KL mu function $\mu(\gamma', \gamma) =$ top coeff of $P_{\gamma',\gamma}$: usually zero.

Encodes W-graph on $\mathbb{B}$; fairly easy to recover any desired family $\{P_{\gamma',\gamma} \mid \gamma' \text{ varying}\}$ (what's needed to write character of one J_γ) by short computation.

Find algorithm to compute only $\mu(\gamma', \gamma)$. This is a version of find algorithm to compute W-graph on $\mathbb{B}$, a problem about which Stembridge has published some results.

Lusztig web page has comments about old papers. Comments about the 1979 KL paper include new algorithm to compute KL polys.

Is the new algorithm comparable in complexity to the old? Can it be extended to the real group setting?

$\mathcal{D}$-modules

Theory of reg holonomic $\mathcal{D}$-mods with respect to a stratif. $\{Z_\alpha \mid \alpha \in A\}$ of alg var Z parametrizes the irrs by pairs $(Z_\alpha, \mathcal{L})$: (stratum,local system). To a pair of such parameters one gets local cohomological data for perverse sheaves. In the case of flag varieties, this is what KL theory can compute.

(Much) more elem idea: characteristic cycle $=$ int comb of conormal bdles $N_{Z_\alpha}(Z)$ to each perverse sheaf. Seems nobody knows how to compute char cycles (perhaps in terms of KL/perverse sheaf data). Case of flag vars could be a good place to try to remedy this.

Nilpotent coadjt orbits

There are many natural ways to attach nilpotent coadjt orbits to $(\mathfrak{g}, K)$-modules, including char cycles as above. One obstruction to finding nice algorithms to compute these is lack of parametrization of nilpotent coadjt orbits by root datum/Weyl gp computations.

Same problem for nilp orbits of $K(\mathbb{C})$ on $[\mathfrak{g}/\mathfrak{k}]^*$; computation should be same level of difficulty as `atlas` computation of `kgb`.

Taking moment map image of conormal bundle defines a natural surjection

$$K(\mathbb{C})\backslash\mathcal{B} \to \text{nilp orbs of } K(\mathbb{C}) \text{ on } [\mathfrak{g}/\mathfrak{k}]^*$$

Find root datum algorithm for the fibers.

Iwahori Hecke algebras and real groups

Suppose G split over $\mathbb{F}_q$.

1. Convolution alg $\mathcal{H}$ of $B(\mathbb{F}_q)$-biinvt fns on $G(\mathbb{F}_q)$ is naturally Iwahori Hecke alg for W specialized to q. Same conv alg is $\mathrm{End}_{G(\mathbb{F}_q)}$(fns on $G(\mathbb{F}_q/B(\mathbb{F}_q))$).

These facts explain Hecke algs control reps of $G(\mathbb{F}_q)$. Works almost as well for reps of p-adic groups.

For real groups, conns with Iwahori Hecke algs are more subtle and indirect. Fixing this might help explain Barbasch-Ciubotaru results comparing real and p-adic groups.

KL theory $\rightsquigarrow$ action of Iwahori Hecke alg of W on free $\mathbb{Z}[q, q^{-1}]$ with basis $\mathbb{B}$ (block of regular integral infl char). See previous problem, and in the same direction enlarge $\mathbb{B}$ by Zuckerman transl, enlarge Hecke alg to affine Hecke alg.

Chevalley homomorphisms

Slide "Why cohomology can identify a module" uses a Chevalley homomorphism

$$\xi\colon R(\mathfrak{g}, L_K)^{\mathfrak{k}, L_K} \to R(\mathfrak{l}, L_K)^{L_K}.$$

This is very easy to define. In the setting of the slide, might have been more natural to ask about

$$\widetilde{\xi}\colon R(\mathfrak{g}, K)^K \xrightarrow{???} R(\mathfrak{l}, L_K)^{L_K}.$$

Is there a natural definition of $\widetilde{\xi}$ (leading to a version of the theorem on the slide)?

Case $G = K$ seems to show what the difficulties are.

Proving Jantzen conjectures

KL conjectures seem to be deep for fundamental reasons; probably it's not productive to look for easy proofs.

But reasoning like that by established mathematicians $\rightsquigarrow$ amazing Ph.D. theses from "second rate" universities.

At any rate we *know* the KL conjs. A wonderful aspect of them is that there is a "parity" on irrs of regular infl character so that

1. $J \neq J'$ of irrs same parity $\implies \mathrm{Ext}^1(J, J') = 0$;
2. $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ θ-stable $\implies$ J'_L can appear in $H^r(\mathfrak{u}, J)$ only if the parity of $r - \dim \mathfrak{u}_{\mathfrak{k}}$ $\mathrm{par}(J) - \mathrm{par}(J')$.

Given these deep facts (and more?), give elem pf of the Jantzen conj (proved by Beilinson-Bernstein).

About Jantzen filtration

Verma module I_λ satisfies (almost trivially)

$$H^p(\mathfrak{n}^-, I_\lambda) = \begin{cases} \mathbb{C}_\lambda & (p = 0) \\ 0 & \text{otherwise} \end{cases} \qquad (COHOM)$$

KL conjecture can be formulated as

$$\text{mult of } \mathbb{C}_{\lambda'} \text{ in } H^p(\mathfrak{n}^-, J_\lambda) = \text{coeff of KL poly } P_{\lambda',\lambda} \qquad (KL)$$

Deduce Jantzen conjecture for $\mathcal{O}$ from ($COHOM$), (KL), and homological algebra.

Possibly a hint for how to try this is

Find a common generalization of ($COHOM$) and (KL), perhaps describing cohomology of some subquotients of Jantzen filtration of I_λ.

These problems make sense for $\mathcal{M}(\mathfrak{g}, K)$.

Possible unitarity algorithm

Hope to get from these ideas a computer program; enter

- real reductive Lie group $G(\mathbb{R})$
- general representation π

and ask whether π is unitary.

Program would say either

- π has no invariant Hermitian form, or
- π has invt Herm form, indef on reps μ_1, μ_2 of K, or
- π is unitary, or
- I'm sorry Dave, I'm afraid I can't do that.

Answers to finitely many such questions $\rightsquigarrow$ complete description of unitary dual of $G(\mathbb{R})$.

This would be a good thing.

An inspirational story

I was an undergrad at University of Chicago, learning interesting math from interesting mathematicians.

I left Chicago to work on a Ph.D. with Bert Kostant.

After finishing, I came back to Chicago to visit.

I climbed up to Paul Sally's office. Perhaps not all of you know what an interesting mathematician he is.

I told him what I'd done in my thesis; since it was representation theory, I hoped he'd find it interesting.

He responded kindly and gently, with a question:

"What's it tell you about *UNITARY* representations?"

The answer, regrettably, was, "not much."

So I tried again.