Proof of Cartan's criterion for solvability

February 27, 2015

In class February 26 I presented a proof of

Theorem 0.1 (Cartan)**.** *Suppose V is a finite-dimensional vector space over a field k of characteristic zero, and $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a Lie subalgebra. Write B for the trace form*

$$B(X, Y) =_{def} \operatorname{tr}(XY) \qquad (X, Y \in \mathfrak{g} \subset \mathfrak{gl}(V)).$$

Then $\mathfrak{g}$ is solvable if and only if the radical of $B|_{\mathfrak{g}}$ contains the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$. In other words, $\mathfrak{g}$ is solvable if and only if

$$\operatorname{tr}(XY) = 0 \qquad (X \in [\mathfrak{g}, \mathfrak{g}], Y \in \mathfrak{g}).$$

The proof was complicated and not so clearly presented in class, and the result is not proved in the text; so these notes attempt to reproduce the argument from class, with some details and explanations added.

Proof. First, the easy parts of the argument. Let K be any extension field of k. The homework for March 3 introduces extension of scalars for vector spaces, algebras, linear maps, and bilinear forms. Extension of scalars preserves dimension of vector spaces, and in particular vanishing or non-vanishing of vector spaces. It's very easy to check that some obvious natural maps define isomorphisms

$$[\mathfrak{g}^{(p)}]_K \simeq [\mathfrak{g}_K]^{(p)}.$$

Consequently $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_K$ is solvable. If $T \colon V \to W$ is a linear map, then

$$\ker(T)_K = \ker(T_K).$$

A bilinear form B on V is the same thing as a linear map from V to $\operatorname{Hom}(V, k)$; the radical of B is the kernel of the map. From these two facts, we deduce that

$$[\operatorname{Rad}(B)]_K = \operatorname{Rad}(B_K).$$

Because of these remarks, Cartan's theorem for $\mathfrak{g}$ and k is exactly equivalent to Cartan's theorem for $\mathfrak{g}_K$ and K. Taking for K an algebraic closure of k, we see that

it's enough to prove the theorem assuming k algebraically closed.

We therefore from now on assume that k is algebraically closed.

If $\mathfrak{g}$ is solvable, then Lie's theorem provides a basis $\{e_1, \ldots, e_n\}$ of V in which $\mathfrak{g}$ consists of upper triangular matrices. Therefore $[\mathfrak{g}, \mathfrak{g}]$ consists of upper triangular matrices with zeros on the diagonal. It follows that any XY as in the theorem is upper triangular with zeros on the diagonal, and therefore has trace zero, as we wished to show.

So much for the easy parts. The hard part is to show that *if*

1. k is algebraically closed;
2. V is a finite-dimensional k-vector space;
3. $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a Lie subalgebra; and
4. $\mathrm{tr}(XY) = 0$, all $X \in [\mathfrak{g}, \mathfrak{g}]$ and $Y \in \mathfrak{g}$,

then $\mathfrak{g}$ is solvable.

(This will complete the proof.) Here is the strategy. First, we will prove a *stronger* vanishing condition than 4). For that, define

$$\mathfrak{h} = \{Z \in \mathfrak{gl}(V) \mid \mathrm{ad}(Z)(\mathfrak{g} \subset [\mathfrak{g}, \mathfrak{g}].$$

(It's easy to see that $\mathfrak{h}$ is a Lie algebra containing $\mathfrak{g}$.) Then we will prove

4′. $\mathrm{tr}(XZ) = 0$, all $X \in [\mathfrak{g}, \mathfrak{g}]$ and $Z \in \mathfrak{h}$.

Here's how to deduce 4′) from 4). Suppose $X = [G_1, G_2]$ is a typical generator of $[\mathfrak{g}, \mathfrak{g}]$, with $G_i \in \mathfrak{g}$; and that $Z \in \mathfrak{h}$. Then

$$\mathrm{tr}(XZ) = \mathrm{tr}([G_1, G_2]Z) = \mathrm{tr}([Z, G_1]G_2);$$

the second equality is the invariance of the trace form discussed in class (which follows immediately from $\mathrm{tr}(AB) = \mathrm{tr}(BA)$). In the last term, the first factor $[Z, G_1]$ belongs to $[\mathfrak{g}, \mathfrak{g}]$ by the definition of $\mathfrak{h}$, and the second to $\mathfrak{g}$; so the trace is zero by 4). This proves 4′).

Here is the strategy to use 4′) to prove that $\mathfrak{g}$ is solvable. We will use

Lemma 0.2 (see Humphreys' Lie algebra book, Lemma 4.3). *Suppose V is a finite-dimensional vector space over a field k of characteristic zero, and that*
$$A \subset B \subset \mathfrak{gl}(V)$$
are subspaces. Define
$$\mathfrak{h} = \{Z \in \mathfrak{gl}(V) \mid \mathrm{ad}(Z)(B) \subset A\}.$$
Suppose that $X \in \mathfrak{h}$ has the property that
$$\mathrm{tr}(XZ) = 0 \qquad (Z \in \mathfrak{h}).$$
Then X is a nilpotent linear transformation.

We postpone the proof of the lemma for a moment and complete the proof of Cartan's theorem. Apply the lemma with $A = [\mathfrak{g}, \mathfrak{g}]$, $B = \mathfrak{g}$. Now hypothesis $4'$) and the lemma tell us that

$$[\mathfrak{g}, \mathfrak{g}] \text{ consists of nilpotent linear transformations.}$$

By Engel's theorem, it follows that $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra. Since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is obviously abelian, it follows that $\mathfrak{g}$ is solvable, as we wished to show.

Proof of the lemma. Write $X = s+n$ for the Jordan decomposition of X (so that s and n commute, s is diagonalizable, and n is nilpotent). The Jordan decomposition of $\mathrm{ad}(X)$ is $\mathrm{ad}(X) = \mathrm{ad}(s) + \mathrm{ad}(n)$. This means in particular that $\mathrm{ad}(s)$ can be written as a polynomial in $\mathrm{ad}(X)$ without constant term. It follows that $\mathrm{ad}(s)$ must (like $\mathrm{ad}(X)$) carry B into A, so $s \in \mathfrak{h}$. (In the setting of Cartan's theorem, s *need not* belong to $\mathfrak{g}$; this difficulty is the reason for introducing the bigger Lie algebra $\mathfrak{h}$.)

Because s is semisimple, we can choose a basis for V of eigenvectors:
$$s \cdot e_i = a_i e_i \qquad (i = 1, \ldots, n).$$
We can order the basis so that n is upper triangular with zeros on the diagonal. Our goal is to prove that all the a_i are zero. The method is to find elements $Z \in \mathfrak{h}$ that are *also* diagonalizable with these same eigenvectors:
$$Z \cdot e_i = b_i e_i \qquad (i = 1, \ldots, n).$$
Then nZ is upper triangular with zeros on the diagonal, so $\mathrm{tr}(nZ) = 0$. The vanishing hypothesis in the lemma tells us that
$$0 = \mathrm{tr}(XZ) = \mathrm{tr}(sZ) = \sum (a_i b_i).$$

We want to deduce from such equations that all the a_i are zero.

What makes this difficult is that we cannot choose arbitrary values for b_i, because such linear transformations will not belong to $\mathfrak{h}$. (If we could choose $b_{i_0} = 1$ and $b_i = 0$ for $i \neq i_0$, then we would get $a_{i_0} = 0$ immediately.) The only obviously allowed choice of b_i is a_i (that is, $Z = s$, which we know is in $\mathfrak{h}$). This leads to the equation $\sum a_i^2 = 0$, which is nice but not enough to deduce that the a_i are all zero.

Here is the construction of Z from Humphreys that I explained in class. Let

$$E = \mathbb{Q}\text{-span of } a_1, \ldots, a_n \subset k,$$

a $\mathbb{Q}$-vector space of dimension at most n. We wish to prove that $E = 0$. We'll do this by proving that any $\mathbb{Q}$-linear map

$$f \colon E \to \mathbb{Q}$$

must be zero. So fix such a linear map, and define the linear map Z by

$$Z \cdot e_i = f(a_i) e_i.$$

I claim that Z belongs to $\mathfrak{h}$. (This is one of the things that I did not explain well in class.) To see that, notice first that

$$ad(s) \cdot e_{ij} = (a_i - a_j) e_{ij}, \quad \mathrm{ad}(Z) e_{ij} = (f(a_i) - f(a_j)) e_{ij} = f(a_i - a_j) e_{ij};$$

the last equality uses the $\mathbb{Q}$-linearity of f. Now let r be any polynomial in $tk[t]$ with the property that

$$r(a_i - a_j) = f(a_i - a_j) \qquad (1 \leq i, j \leq n).$$

This requirement specifies values of r at most n^2 different points in k, including a value of 0 at 0; so it can certainly be met by a polynomial without constant term. Comparing the last two displays shows that

$$\mathrm{ad}(Z) = r(\mathrm{ad}(s)).$$

Since the right side evidently carries B into A, the left side does as well; so $Z \in \mathfrak{h}$.

The equation $\mathrm{tr}(XZ) = 0$ is now

$$0 = \sum_{i=1}^{n} a_i f(a_i).$$

The right side is a rational linear combination of the a_i, so it belongs to E. We can therefore apply the linear functional f to it and get

$$0 = \sum_{i=1}^{n} f(a_i)^2.$$

The right side is now a sum of squares of rational numbers; so the conclusion is that $f(a_i) = 0$ for all i. Since the a_i span E, this implies that $f = 0$, as we wished to show. $\square$

With the proof of the lemma, the proof of Cartan's theorem is now complete. $\square$