

# Bilinear Forms

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We may begin our discussion of bilinear forms by looking at a special case that we are already familiar with. Given a vector space  $V$  over a field  $F$ , the dot product between two elements  $X$  and  $Y$  (represented as column vectors whose elements are in  $F$ ) is the map  $V \times V \rightarrow F$  defined by:

$$\langle X, Y \rangle = X^T \cdot Y = x_1y_1 + \dots + x_ny_n$$

The property of the dot product which we will use to generalize to bilinear forms is bilinearity: the dot product is a linear function from  $V$  to  $F$  if one of the elements is fixed.

**Definition** Let  $V$  be a vector space over a field  $F$ . A *bilinear form*  $B$  on  $V$  is a function of two variables  $V \times V \rightarrow F$  which satisfies the following axioms:

$$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w) \tag{1}$$

$$B(fv, w) = fB(v, w) \tag{2}$$

$$B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2) \tag{3}$$

$$B(v, fw) = fB(v, w) \tag{4}$$

When working with linear transformations, we represent our transformation by a square matrix  $A$ . Similarly, given a square matrix  $\hat{B}$ , we may define a bilinear form for all  $v, w \in V$  as follows:

$$B(v, w) = v^T \hat{B}w$$

This form satisfies the axioms because of the distributive laws and the ability to pull out a scalar in matrix multiplication. We can also see that given a bilinear form  $B$ , after fixing a basis the matrix representing  $B$  is unique. Given a basis  $\{b_1, \dots, b_n\}$  of  $V$ , the matrix defined by  $\hat{B}_{i,j} = B(b_i, b_j)$  is the unique matrix representing the form  $B$ . To see this, we can actually

compute the value of the bilinear form for arbitrary  $v, w \in V$ . Since  $\{b_i\}$  is a basis for  $V$ , we have  $v = \sum_i v_i b_i$  and  $w = \sum_i w_i b_i$ , where  $v_i, w_i \in F$ . Then

$$B(v, w) = B\left(\sum_i v_i b_i, \sum_j w_j b_j\right) = \sum_{i,j} v_i w_j B(b_i, b_j) = v^T \hat{B} w$$

where  $v$  and  $w$  are represented as column matrices whose elements are  $v_i$  and  $w_i$  respectively.

Since the matrix of a bilinear form is dependent on the choice of basis, we might wonder how this matrix is affected by a change of basis. Given our original basis  $\{b_i\}$  and a new basis  $\{c_i\}$ , we can express our new basis vectors as linear combinations of the old ones:  $c_j = \sum_i p_{i,j} b_i$  where  $P$  is our invertible change of basis matrix. To compute the entries for the new matrix  $\hat{B}'$  for the bilinear form, we compute the values of the form on the new basis vectors:  $\hat{B}'_{i,j} = B(c_i, c_j) = B\left(\sum_k p_{k,i} b_k, \sum_l p_{l,j} b_l\right) = \sum_{k,l} p_{k,i} p_{l,j} B(b_k, b_l) = \{P^T \hat{B} P\}_{i,j}$ . So the new matrix for the bilinear transform is  $\hat{B}' = P^T \hat{B} P$

Given a bilinear form, we would like to be able to classify it by just a single element of our field  $F$  to be able to read certain properties of the form. The determinant of the matrix  $\hat{B}$  of our form would be a good mapping to use except for the fact that the matrix of the form, and hence the determinant associated with the form, is dependent on the choice of basis. Since we know how the matrix changes when the basis is changed, we can look at how the determinant changes. If we change bases with associated matrix  $P$ , then the new matrix is  $P^T \hat{B} P$  so that the new determinant is  $\det(P^T \hat{B} P) = \det(P^T) \det(\hat{B}) \det(P) = \det(P)^2 \det(\hat{B})$  (since  $\det(P) = \det(P^T)$ ). Since any invertible matrix can serve as a change of basis matrix, and we can construct an invertible matrix to have any nonzero determinant we wish (make it diagonal with all 1s except one diagonal element set to be the desired determinant), the set of determinants that we can associate with a given bilinear form are all multiples of each other by a square element of  $F$ , and conversely given any square element of  $F$ , the set must be closed under multiplication by this element. So we can define a new quantity, called the *discriminant* of a bilinear form, to be associated with the set of determinants we can obtain from matrices associated with the bilinear form.

To be more precise, we define the subgroup of square elements in the multiplicative group of  $F$ :  $F^{\times 2} = \{f^2 : f \in F^*\}$ . We can see that this is a normal subgroup in  $F^*$  because conjugating a square by an element produces a new element which is also a square:  $f a^2 f^{-1} = (f a f^{-1})^2$ . So the discriminant is actually a mapping from the set of bilinear forms on  $V$  to the quotient group we produce using the normal subgroup of squares (or possibly the element 0 which is not in the multiplicative group). The discriminant is defined in such a way that it is independent of the choice of basis.

**Definition** Let  $B$  be a bilinear form on a vector space  $V$  and  $\hat{B}$  be a matrix associated with

the form. The *discriminant* of  $B$  is defined to be

$$\text{discr}(B) = \begin{cases} 0 & \text{if } \det \hat{B} = 0 \\ \det \hat{B} F^{\times 2} \in F^*/F^{\times 2} & \text{otherwise.} \end{cases}$$

We call a bilinear form  $B$  *nondegenerate* if the discriminant of  $B$  is nonzero. To be able to apply the properties of the discriminant and nondegeneracy, we must first understand orthogonality. Given vectors  $v, w \in V$  we say that  $v$  is *orthogonal* to  $w$  (denoted  $v \perp w$ ) if  $B(v, w) = 0$ . We would like to describe the vectors in a vector space that are orthogonal to everything else in that vector space. This set of vectors is referred to as the radical. Since orthogonality is not necessarily a commutative relation, we need to be more specific. Given a vector space  $V$  and a bilinear form  $B$ , we define the *left* and *right radicals* as follows:

$$\text{rad}_L(V) = \{v \in V : B(v, w) = 0, \forall w \in V\}$$

$$\text{rad}_R(V) = \{v \in V : B(w, v) = 0, \forall w \in V\}$$

**Proposition 0.1** *A bilinear form  $B$  on a vector space  $V$  is nondegenerate  $\iff \text{rad}_R(V) = 0 \iff \text{rad}_L(V) = 0$*

**Proof** I will only prove the first equivalence because the proof for the second is similar.

$\Rightarrow$  Assume there is a nonzero element  $X$  of  $\text{rad}_R(V)$  and  $B$  is nondegenerate and fix a bases  $\{b_1, \dots, b_n\} \Rightarrow \hat{B}X \neq 0 \Rightarrow$  there exists some  $w \in V$  such that  $B(w, X) \neq 0 \Rightarrow X \notin \text{rad}_R(V)$  which is a contradiction.

$\Leftarrow$  Assume  $B$  not nondegenerate and fix a basis  $\{b_1, \dots, b_n\} \Rightarrow \det(\hat{B}) = 0 \Rightarrow$  there is a nontrivial solution to the matrix equation  $AX = 0 \Rightarrow w^T \hat{B}X = 0$  for all  $w \in V \Rightarrow X \in \text{rad}_R(V) \Rightarrow \text{rad}_R(V)$  has a nonzero element.

While in the definition of a radical we include elements of  $V$  that are orthogonal to all other elements of  $V$ , we may be more specific and seek only elements of  $V$  that are orthogonal to all elements of some subset  $S$  of  $V$ :

$$\perp_L(S) = \{v \in V : B(v, w) = 0, \forall w \in S\}$$

$$\perp_R(S) = \{v \in V : B(w, v) = 0, \forall w \in S\}$$

It would be nice if we did not have to deal with all of the distinguishing between left and right orthogonality. A bilinear form  $B$  such that  $B(v, w) = 0 \iff B(w, v) = 0$  for all  $v, w \in V$  is called *reflexive*. Given a reflexive bilinear form and a subset  $S$  of  $V$ , we may write  $\perp_L(S) = \perp_R(S) = S^\perp$ . If  $W$  is a subspace of  $V$  then we call  $W^\perp$  the *orthogonal complement* of  $W$ . We define the *radical* of a subspace  $W$  of  $V$  to be  $\text{rad}W = W \cap W^\perp$  and call  $W$  a *nondegenerate subspace* if  $\text{rad}W = 0$ .

**Proposition 0.2** *Let  $B$  be a reflexive bilinear form on a vector space  $V$ , and  $W$  be a non-degenerate subspace of  $V$ . Then  $V = W \oplus W^\perp$ .*

**Proof** We know that  $W \cap W^\perp = 0$  because  $W$  is nondegenerate. So all we need to show is that together  $W$  and  $W^\perp$  span  $V \iff \dim V = \dim W + \dim W^\perp$ . Let  $n = \dim V$  and  $k = \dim W$  and fix a basis  $\{v_1, \dots, v_k\}$  of  $W$  and extend to a basis of  $V$  with  $\{v_{k+1}, \dots, v_n\}$ . Given arbitrary  $v \in W^\perp$ , we may write  $v = c_1 v_1 + \dots + c_n v_n$  and  $v \in W^\perp \Rightarrow B(v_i, v) = 0 \forall 1 \leq i \leq k \Rightarrow \sum_j B(v_i, v_j) c_j = 0 \Rightarrow \sum_j \hat{B}(i, j) c_j = 0 \Rightarrow v$  is in the null space of a  $k \times n$  matrix which is the top  $k$  rows of the matrix  $\hat{B} \Rightarrow \dim W^\perp \geq n - k \Rightarrow \dim W + \dim W^\perp = \dim W \oplus \dim W^\perp \geq n + n - k = n \Rightarrow W \oplus W^\perp = V$ .

**Definition** A bilinear form  $B$  on a vector space  $V$  is called *symmetric* if  $B(v, w) = B(w, v)$  for all  $w, v \in V$

We can see that the matrix  $\hat{B}$  of a symmetric bilinear form must itself be symmetric by taking standard basis vectors  $e_i$  and  $e_j$ :

$$B(e_i, e_j) = B(e_j, e_i) \iff e_i^T \hat{B} e_j = e_j^T \hat{B} e_i \iff \hat{B}_{i,j} = \hat{B}_{j,i} \iff \hat{B} = \hat{B}^T$$

**Definition** A bilinear form  $B$  on a vector space  $V$  is called *alternate* (or *skew-symmetric* if  $\text{char } F \neq 2$ ) if  $B(v, v) = 0$  for all  $w, v \in V$

If we have an alternate form  $B$  and take arbitrary  $v, w \in V$  then  $0 = B(v + w, v + w) = B(v, v) + B(v, w) + B(w, v) + B(w, w) = B(v, w) + B(w, v) \rightarrow B(v, w) = -B(w, v)$ . If the character of  $F$  is not 2, then the implication is reversible and so we can replace the condition for an alternate form with the condition on a skew-symmetric form, namely that  $B(v, w) = -B(w, v)$  for all  $w, v \in V$ . We also know that the matrix of an alternate form must itself be alternate. The same method used for symmetric forms shows that such a matrix  $\hat{B}$  must satisfy  $\hat{B} = -\hat{B}^T$ .