Associative algebras, Lie algebras, and bilinear forms

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1 Introduction

The most basic and important example of a Lie group is the group $GL(n,\mathbb{R})$ of invertible $n \times n$ matrices. This group is very closely related to the associative algebra $M(n,\mathbb{R})$ of *all* $n \times n$ matrices. In particular, the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ is just given by commutator in the associative algebra $M(n,\mathbb{R})$.

These notes examine orthogonal and symplectic groups from a similar point of view. The groups are defined as subgroups of $GL(n,\mathbb{R})$ preserving some bilinear forms. We want to offer a related perspective on them, as related to some associative algebras. In the case of symplectic groups, the associative algebra is a *Weyl algebra* of polynomial coefficient differential operators. In the case of orthogonal groups, it is a *Clifford algebra.* The two theories are in many respects exactly parallel; if my TEX skills were greater, I would have written these notes in two columns, doing the two cases side by side. As it is, I need to put one first. The Weyl algebra is infinite-dimensional, and for that reason a bit scarier; but it consists of differential operators, and so is more familiar. The Clifford algebra is finite-dimensional, and so technically simpler, but much less familiar.

I will write about the symplectic case first; you can read in whichever order you prefer.

2 Symplectic groups and Weyl algebras

We begin with a finite-dimensional real vector space V, endowed with a *nondegenerate* skew-symmetric bilinear form

$$\omega\colon V \times V \to \mathbb{R}, \qquad \omega(v,w) = -\omega(w,v). \tag{2.1a}$$

The *symplectic group* is

$$Sp(V) = \{g \in GL(V) \mid \omega(gv, gw) = \omega(v, w)\}. \tag{2.1b}$$

Its Lie algebra is

$$\mathfrak{sp}(V) = \{X \in \mathfrak{gl}(V) \mid \omega(Xv, w) + \omega(v, Xw) = 0\}. \tag{2.1c}$$

It is an elementary exercise to show that V has a *symplectic basis*

$$(p_1, \ldots, p_n, q_1, \ldots, q_n) : \tag{2.1d}$$

the defining properties are

$$\omega(p_i, p_j) = 0, \quad \omega(q_k, q_\ell) = 0, \quad \omega(p_i, q_k) = \delta_{ik}. \tag{2.1e}$$

In particular, V must be even-dimensional. In this basis,

$$\mathfrak{sp}(\mathbb{R}^{2n}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A = -{}^tD,\ B = {}^tB,\ C = {}^tC \right\}; \tag{2.1f}$$

here A, B, C, and D are $n \times n$ matrices.

We want to make an associative algebra related to V and to ω. From V one can construct the *tensor algebra*

$$T^m(V) = \underbrace{V \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V}_{m \text{ copies}}, \qquad T(V) = \sum_{m=0}^{\infty} T^m(V). \tag{2.1g}$$

Here we define

$$T^0(V) = \mathbb{R}, \tag{2.1h}$$

which contains the multiplicative identity element 1 for $T(V)$. The algebra $T(V)$ can be thought of as the free associative algebra generated by V. The group $GL(V)$ acts on $T(V)$ by algebra automorphisms. The algebra $T(V)$ is $\mathbb{Z}$-graded, and the action of $GL(V)$ preserves this grading.

We are going to impose relations on $T(V)$ using the symplectic form ω. The *Weyl algebra* $A(V)$ is by definition the quotient of $T(V)$ by the ideal generated by the elements

$$v \otimes w - w \otimes v - \omega(v, w) \qquad (v, w \in V). \tag{2.1i}$$

Because $\otimes$ is the multiplication in $T(V)$, we can write these elements as

$$vw - wv - \omega(v, w) \qquad (v, w \in V). \tag{2.1j}$$

In terms of the symplectic basis $(p_1, \ldots, q_n)$, these relations become

$$p_i p_j = p_j p_i, \quad q_k q_\ell = q_\ell q_k, \qquad p_i q_k - q_k p_i = \delta_{ik}. \tag{2.1k}$$

These are the *canonical commutation relations* of quantum mechanics.

Physicists often prefer to put a factor like $\sqrt{-1}\hbar$ in front of p_i, so that the commutation relation gets this factor in front of it. The factor is mathematically helpful in reminding us that (sometimes) we should think of the Weyl algebra as almost commutative: the commutators are small. But since the speed of light is one anyway, I will keep life mathematically simpler and omit the factor.

Our definition is

$$A(V) = T(V)/\langle vw - wv - \omega(v,w) \mid v, w \in V \rangle; \tag{2.1l}$$

the angle brackets are meant to indicate "ideal generated by." The commutation relations are not homogeneous: they belong to $T^2(V) \oplus T^0(V)$ rather than to one degree separately. As a consequence, $A(V)$ does *not* inherit the graded algebra structure from $T(V)$. The filtered algebra structure survives:

$$A^{\leq p}(V) = \mathrm{im}\left(\sum_{m=0}^{p} T^m(V)\right), \qquad A^{\leq p} A^{\leq q} \subset A^{\leq p+q}. \tag{2.1m}$$

In addition, the relations are *even*, so $A(V)$ *does* inherit from $T(V)$ a $\mathbb{Z}/2\mathbb{Z}$ grading:

$$A^{\mathrm{even}}(V) = \mathrm{im}\left(\sum_m T^{2m}(V)\right), \quad A^{\mathrm{odd}}(V) = \mathrm{im}\left(\sum_m T^{2m+1}(V)\right). \tag{2.1n}$$

That this is a grading means that

$$A^{\mathrm{even}}(V) A^{\mathrm{odd}}(V) \subset A^{\mathrm{odd}}(V) \tag{2.1o}$$

and so on. Multiplication in the tensor algebra adds degrees, and so does taking commutators: if s is a tensor of degree p and t a tensor of degree q, then st has degree exactly $p+q$, and $st - ts$ also has degree exactly $p+q$ (although the commutator might be zero; zero is declared to be homogeneous of any degree, so the the tensors of a fixed degree can be a vector space). But in the Weyl algebra, the nature of the defining relations guarantees that commutator *lowers* degrees:

$$a \in A^{\leq p}, \quad b \in A^{\leq q} \implies ab - ba \in A^{\leq p+q-1}. \tag{2.1p}$$

Proposition 2.2. *Suppose we are in the setting* (2.1).

1. *The graded action of $Sp(V) \subset GL(V)$ on $T(V)$ descends to the Weyl algebra $A(V)$, identifying $Sp(V)$ as the group of filtration-preserving algebra automorphisms of $A(V)$.*

2. *The associated graded algebra*
$$\mathrm{gr}(A(V)) = \sum_{m=0}^{\infty} A^{\leq m}(V)/A^{\leq m-1}(V)$$
is isomorphic to the symmetric algebra $S(V)$.

3. *The Lie algebra $\mathfrak{sp}(V)$ may be identified with the filtration-preserving derivations of $A(V)$; equivalently, with linear maps*
$$X\colon V \to V, \qquad \omega(Xv, w) + \omega(v, Xw) = 0.$$

4. *Every such derivation is commutator with an element of*
$$A^{\leq 2, even} \simeq S^2(V) \oplus \mathbb{R}.$$

5. *The image of $S^2(V)$ (identified with symmetric 2-tensors and then mapped to $A^{\leq 2}$) is closed under commutator, and so identified with $\mathfrak{sp}(V)$.*

6. *Any module M for the associative algebra $A(V)$ is automatically (by restriction to $S^2(V)$) a module for the Lie algebra $\mathfrak{sp}(V)$.*

Proof. The assertions in (1) are more or less obvious from the definition of $A(V)$. The surjective map
$$S(V) \to \mathrm{gr}\, A(V) \tag{2.3a}$$
in (2) exists because of (2.1p). That it is an isomorphism can be proven directly, but I prefer to prove it either as part of Theorem 2.4 or as part of Theorem 2.6 below.

For (3), the action of $Sp(V)$ may be studied in the finite-dimensional subspaces $A^{\leq p}(V)$, so finite-dimensional linear algebra (including the exponential map) can be used. The derivative of the definition of automorphism is the definition of derivation: more precisely, $\exp(tX)$ is an automorphism for all t if and only if X is a derivation. Now (3) follows.

For (4), if $a \in A^{\leq 2,\text{even}}$ and

$$v \in A^{\leq 1,\text{odd}} = V, \tag{2.3b}$$

then $[a, v]$ is odd of degree at most 2. As a consequence of (2),

$$A^{\leq 2,\text{odd}} = A^{\leq 1,\text{odd}} \simeq V. \tag{2.3c}$$

This shows that commutator with a preserves V, and therefore is filtration preserving. We have therefore defined a Lie algebra homomorphism

$$A^{\leq 2,\text{even}} \to \mathfrak{sp}(V), \qquad a \mapsto (v \mapsto [a, v]). \tag{2.3d}$$

Using the canonical commutation relations, one can compute this map explicitly in a symplectic basis. To simplify the notation, I will do this just for $n = 1$, with a symplectic basis (p, q). Then (using $pq - qp = 1$)

$$\begin{aligned} [p^2, q] &= p^2q - qp^2 \\ &= p(qp + 1) - qp^2 = pqp - qp^2 + p \\ &= (qp + 1)p - qp^2 + p = 2p. \end{aligned}$$

Similarly,

$$[p^2, p] = 0, \qquad [q^2, p] = -2q, \qquad [q^2, q] = 0.$$

Finally,

$$[pq + qp, p] = -2p, \qquad [pq + qp, q] = 2q.$$

The map (2.3d) is therefore

$$p^2 \mapsto \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad q^2 \mapsto \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \quad pq + qp \mapsto \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}. \tag{2.3e}$$

The three matrices here span $\mathfrak{sp}(\mathbb{R}^2)$ (see (2.1f)), proving that (2.3d) is an isomorphism when $\dim V = 2$. The general case is identical.

I don't know a reasonable proof of the assertion in (5); one can compute all the commutators by hand, but this is not very nice. Part (6) is immediate (being true for any Lie subalgebra of any associative algebra). $\square$

Theorem 2.4. *The Weyl algebra may be identified with the algebra of polynomial coefficient differential operators in the n variables $q_1, \ldots, q_n$: the identification is*

$$p_i \mapsto \frac{\partial}{\partial q_i}, \qquad q_k \mapsto \textit{multiplication by } q_k.$$

In this identification, $A^{\leq p}(V)$ corresponds to differential operators of total degree (adding degrees of differentiation and degrees of polynomial coefficients) at most p.

Proof. Write P for the algebra of polynomial coefficient differential operators. To get an algebra surjection

$$A(V) \to P, \tag{2.5}$$

one just has to verify that the canonical commutation relations (2.1k) are satisfied by the corresponding differential operators. The main one is

$$\frac{\partial(q_k f)}{\partial q_i} - q_k \frac{\partial f}{\partial q_i} = \delta_{ik} f,$$

which is the Leibnitz rule.

Because the elements $q^\alpha \frac{\partial^\beta}{\partial q^\beta}$ are a basis of P, we see that the map from $S(V)$ (with basis $p^\beta q^\alpha$) to $\operatorname{gr} P$ (composing the surjection (2.3a) with gr of the surjection (2.5)) is an isomorphism. It follows (2.3a) and (2.5) are both isomorphisms. □

In the setting of orthogonal groups, the best analogues of symplectic bases (which involve hyperbolic planes) are not always available without complexification; and even then orthogonal matters are a little more complicated. We will therefore give a cruder version of Theorem 2.4 that does not rely on symplectic bases.

Theorem 2.6. *Suppose V is any vector space. The algebra $P(V)$ of polynomial coefficient differential operators on functions on V has the following generators. To each element $v \in V$ is attached the first-order differential operator $\partial(v)$ which is the directional derivative in the direction v:*

$$[\partial(v)f](x) = \frac{d}{dt} f(x + tv)|_{t=0}.$$

To each linear functional λ on V is attached the zeroth-order differential operator $m(\lambda)$ which is multiplication by the function λ:

$$[m(\lambda)f](x) = \lambda(x)f(x).$$

All the operators $\partial(v)$ commute with each other, as do all the operators $m(\lambda)$. We have

$$[\partial(v), m(\lambda)] = \lambda(v),$$

multiplication by the constant $\lambda(v)$.

Suppose now that V *is a symplectic vector space. To each* $v \in V$ *corresponds the linear functional*

$$\lambda_v : V \to \mathbb{R}, \qquad \lambda_v(x) = \omega(x, v).$$

The Weyl algebra $A(V)$ *may be embedded in* $P(V)$ *by sending* v *to the first-order differential operator*

$$D(v) = \partial(v) + \lambda_v/2.$$

Proof. The linear map D defines (by the universality of tensor products) an algebra map $T(V) \to P(V)$, which we continue to denote by D. We compute

$$\begin{aligned}
[D(v), D(w)] &= [\partial(v), \lambda_w/2] - [\partial(w), \lambda_v/2] \\
&= \lambda_w(v)/2 - \lambda_v(w)/2 = \omega(v, w)/2 - \omega(w, v)/2 \\
&= \omega(v, w).
\end{aligned}$$

This says that D vanishes on the defining ideal of $A(V)$ (see (2.1l)), so it descends to an algebra homomorphism

$$D : A(V) \to P(V).$$

Clearly D sends the filtration on $A(V)$ to the degree filtration on differential operators. The associated graded map, composed with the surjection of (2.3a), is the isomorphism

$$S(V) \to \text{constant coefficient differential operators on } V.$$

It follows first that D is an inclusion, and second that the surjection of (2.3a) is an isomorphism. $\square$

3 Orthogonal groups and Clifford algebras

We begin with a finite-dimensional real vector space V, endowed with a *nondegenerate* symmetric bilinear form

$$Q : V \times V \to \mathbb{R}, \qquad Q(v, w) = Q(w, v).$$

The *orthogonal group* is

$$O(V) = \{g \in GL(V) \mid Q(gv, gw) = Q(v, w)\}.$$

Its Lie algebra is

$$\mathfrak{o}(V) = \{X \in \mathfrak{gl}(V) \mid Q(Xv, w) + Q(v, Xw) = 0\}.$$

It is an elementary exercise to show that V has an *orthogonal basis*

$$(e_1, \ldots, e_{p+q}), \qquad Q(e_r, e_s) = \epsilon_r \delta_{rs};$$

here

$$\epsilon_r = \begin{cases} 1 & 1 \le r \le p, \\ -1 & p+1 \le r \le p+q \end{cases} \tag{3.1a}$$

(It would be mathematically better to call an orthogonal basis any in which the basis vectors are mutually orthogonal, but have arbitrary lengths not necessarily equal to ± 1. With this more general definition, a nondegenerate quadratic form over any field of characteristic not two has an orthogonal basis; and almost all of the discussion below carries through with just a bit more notation.) In this basis,

$$\mathfrak{o}(p, q) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A = -{}^t A,\ B = {}^t C,\ D = -{}^t D \right\}; \tag{3.1b}$$

here A is $p \times p$, B is $p \times q$, C is $q \times p$, and D is $q \times q$.

We want to make an associative algebra related to V and to Q. As in the symplectic case we begin with the tensor algebra

$$T^m(V) = \underbrace{V \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V}_{m \text{ copies}}, \qquad T(V) = \sum_{m=0}^{\infty} T^m(V).$$

We are going to impose relations on $T(V)$ using the symmetric form Q. The *Clifford algebra* $C(V)$ is by definition the quotient of $T(V)$ by the ideal generated by the elements

$$v \otimes w + w \otimes v + 2Q(v, w) \qquad (v, w \in V).$$

Because $\otimes$ is the multiplication in $T(V)$, we can write these elements as

$$vw + wv + 2Q(v, w) \qquad (v, w \in V).$$

In terms of the orthogonal basis $(e_1, \ldots, e_{p+q})$, these relations become

$$e_r e_s = -e_s e_r \quad (1 \le r \ne s \le p+q), \qquad e_r^2 = -\epsilon_r. \tag{3.1c}$$

Here are some examples. If $p = 1$ and $q = 0$, the Clifford algebra is

$$C(1,0) = \mathbb{R}[e_1]/\langle e_1^2 = -1\rangle \simeq \mathbb{C},$$

the complex numbers (with $e_1 = i$). If $p = 2$ and $q = 0$, the Clifford algebra is

$$C(2,0) = \mathbb{R}[e_1, e_2]/\langle e_r^2 = -1,\ e_1e_2 = -e_2e_1\rangle.$$

This is the algebra of quaternions, with $e_1 = i$, $e_2 = j$, and $e_1e_2 = k$.

Our general definition is

$$C(V) = T(V)/\langle vw + wv + 2Q(v,w) \mid v, w \in V\rangle; \tag{3.1d}$$

the angle brackets are meant to indicate "ideal generated by." The commutation relations are not homogeneous: they belong to $T^2(V) \oplus T^0(V)$ rather than to one degree separately. As a consequence, $C(V)$ does *not* inherit the graded algebra structure from $T(V)$. The filtered algebra structure survives:

$$C^{\leq p}(V) = \mathrm{im}\left(\sum_{m=0}^{p} T^m(V)\right), \qquad C^{\leq p}C^{\leq q} \subset C^{\leq p+q}. \tag{3.1e}$$

In addition, the relations are *even*, so $C(V)$ *does* inherit from $T(V)$ a $\mathbb{Z}/2\mathbb{Z}$ grading:

$$C^{\mathrm{even}}(V) = \mathrm{im}\left(\sum_m T^{2m}(V)\right), \quad C^{\mathrm{odd}}(V) = \mathrm{im}\left(\sum_m T^{2m+1}(V)\right).$$

That this is a grading means that

$$C^{\mathrm{even}}(V)C^{\mathrm{odd}}(V) \subset C^{\mathrm{odd}}(V) \tag{3.1f}$$

and so on. Multiplication in the tensor algebra adds degrees, and so does taking commutators or anticommutators: if s is a tensor of degree p and t a tensor of degree q, then st has degree exactly $p + q$, and $st \pm ts$ also has degree exactly $p + q$ (although it might be zero; zero is declared to be homogeneous of any degree, so the the tensors of a fixed degree can be a vector space).

A $\mathbb{Z}$-graded algebra L is called *anticommutative* if

$$\ell_r\ell_s = (-1)^{rs}\ell_s\ell_r \qquad (\ell_r \in L_r,\ \ell_s \in L_s).$$

Just as the canonical commutation relations say that the Weyl algebra is "approximately" commutative, the Clifford relations say that the Clifford algebra is approximately anticommutative:

$$a \in C^{\leq r}, \quad b \in C^{\leq s} \implies ab = (-1)^{rs}ba + C^{\leq r+s-1}. \tag{3.1g}$$

Proposition 3.2. *Suppose we are in the setting* (3.1).

1. *The graded action of $O(V) \subset GL(V)$ on $T(V)$ descends to the Clifford algebra $C(V)$, identifying $O(V)$ as the group of filtration-preserving algebra automorphisms of $C(V)$.*

2. *The associated graded algebra*

$$\mathrm{gr}(C(V)) = \sum_{m=0}^{\infty} C^{\leq m}(V)/C^{\leq m-1}(V)$$

   is isomorphic to the (graded anticommutative) algebra $\bigwedge(V)$.

3. *The Lie algebra $\mathfrak{o}(V)$ may be identified with the filtration-preserving derivations of $C(V)$; equivalently, with linear maps*

$$X\colon V \to V, \qquad Q(Xv, w) + Q(v, Xw) = 0.$$

4. *Every such derivation is commutator with an element of*

$$C^{\leq 2, even} \simeq \textstyle\bigwedge^2(V) \oplus \mathbb{R}.$$

5. *The image of $\bigwedge^2(V)$ (identified with antisymmetric 2-tensors and then mapped to $C^{\leq 2}$) is closed under commutator, and so identified with $\mathfrak{o}(V)$.*

6. *Any module S for the associative algebra $C(V)$ is automatically (by restriction to $\bigwedge^2(V)$) a module for the Lie algebra $\mathfrak{o}(V)$.*

Proof. The assertions in (1) are more or less obvious from the definition of $C(V)$. The surjective map

$$\textstyle\bigwedge(V) \to \mathrm{gr}\, C(V) \tag{3.3a}$$

in (2) exists because of (3.1g). That it is an isomorphism can be proven directly, but I prefer to prove it as part of Theorem 3.4 below. At any rate we now know that

$$\dim C(V) \leq \dim \textstyle\bigwedge(V) = 2^{\dim V}.$$

Part (3) is technically simpler than the corresponding statement for the Weyl algebra because $C(V)$ is finite-dimensional. The derivative of the definition of automorphism is the definition of derivation: more precisely (given a linear transformation X of a finite-dimensional real algebra), $\exp(tX)$ is an automorphism for all t if and only if X is a derivation. Now (3) follows.

For (4), if $a \in C^{\leq 2,\text{even}}$ and

$$v \in C^{\leq 1,\text{odd}} = V, \tag{3.3b}$$

then $[a, v]$ is odd of degree at most 2. As a consequence of (2),

$$C^{\leq 2,\text{odd}} = C^{\leq 1,\text{odd}} \simeq V. \tag{3.3c}$$

This shows that commutator with a preserves V, and therefore is filtration preserving. We have therefore defined a Lie algebra homomorphism

$$C^{\leq 2,\text{even}} \to \mathfrak{o}(V), \qquad a \mapsto (v \mapsto [a, v]). \tag{3.3d}$$

Using the Clifford commutation relations (3.1c), one can compute this map explicitly in an orthogonal basis. To simplify the notation, I will do this just for $n = 2$, with an orthogonal basis (e_1, e_2). Then (using $e_1e_2 + e_2e_1 = 0$ and $e_j^2 = \epsilon_j$)

$$\begin{aligned} [e_1e_2, e_1] &= e_1e_2e_1 - e_1^2e_2 \\ &= -2e_1^2e_2 = -2\epsilon_1e_2 \end{aligned}$$

Similarly,

$$[e_1e_2, e_2] = 2\epsilon_2e_2,$$

The map (3.3d) is therefore

$$e_1e_2 \mapsto \begin{pmatrix} 0 & 2\epsilon_2 \\ -2\epsilon_1 & 0 \end{pmatrix}. \tag{3.3e}$$

This matrix spans $\mathfrak{o}(\mathbb{R}^2)$ (see (3.1b)), proving that (3.3d) is an isomorphism when $\dim V = 2$. The general case is identical.

I don't know a reasonable proof of the assertion in (5); one can compute all the commutators by hand, but this is not very nice. Part (6) is immediate (being true for any Lie subalgebra of any associative algebra). □

Theorem 3.4. *Suppose V is any vector space. Recall that the exterior algebra $\bigwedge(V)$ is the quotient of $T(V)$ by the ideal generated by elements $v \otimes v$. We consider an algebra $E(V)$ of linear transformations of $\bigwedge V$ with the following generators. To each element $v \in V$ is attached the operator $m(v)$ (raising degree by 1) which is multiplication by v:*

$$m(v)(x_1 \wedge \cdots \wedge x_r) = v \wedge x_1 \wedge \cdots \wedge x_r.$$

Because of the definition of the exterior algebra,

$$m(v)^2 = 0.$$

To each linear functional λ on V is attached the operator $\iota(\lambda)$ (lowering degree by 1) which is interior product with λ:

$$\iota(\lambda)(x_1 \wedge \cdots \wedge x_r) = \sum_{i=1}^{r} (-1)^{i-1} \lambda(x_i)(x_1 \wedge \cdots \widehat{x_i} \cdots \wedge x_r).$$

(The notation means that the factor x_i on the right is omitted.) Not quite as obviously as for $m(v)$, but pretty easily,

$$\iota(\lambda)^2 = 0.$$

All the operators $m(v)$ anticommute with each other, as do all the operators $\iota(\lambda)$. We have

$$\iota(\lambda)m(v) + m(v)\iota(\lambda) = \lambda(v),$$

multiplication by the constant $\lambda(v)$.

Suppose now that V is an orthogonal vector space. To each $v \in V$ corresponds the linear functional

$$\lambda_v \colon V \to \mathbb{R}, \qquad \lambda_v(x) = Q(x, v).$$

The Clifford algebra $C(V)$ may be embedded in $E(V)$ by sending v to the operator

$$\delta(v) = m(v) - \iota(\lambda_v).$$

Proof. The linear map δ defines (by the universality of tensor products) an algebra map $T(V) \to E(V)$, which we continue to denote by δ. We compute

$$\begin{aligned}
\delta(v)\delta(w) + \delta(w)\delta(v) &= -m(v)\iota(\lambda_w) - \iota(\lambda_w)m(v) - m(w)\iota(\lambda_v) - \iota(\lambda_w)m(v) \\
&= -\lambda_w(v) - \lambda_v(w) = -Q(v, w) - Q(w, v) \\
&= -2Q(v, w).
\end{aligned}$$

This says that δ vanishes on the defining ideal of $C(V)$ (see (3.1d)), so it descends to an algebra homomorphism

$$\delta \colon C(V) \to E(V).$$

Clearly δ sends the filtration on $C(V)$ to the degree filtration on $E(V)$ (how much does the linear transformation increase degree in the exterior algebra).

The associated graded map, composed with the surjection of (3.3a), is the isomorphism

$$\bigwedge(V) \to \text{wedge-with-something linear operators on } \bigwedge(V).$$

It follows first that δ is an inclusion, and second that the surjection of (3.3a) is an isomorphism. $\square$

4 Orthogonal groups are not simply connected

Suppose $n \geq 3$. I proved in class that

$$|\pi_1(SO(n))| \leq 2. \tag{4.1a}$$

I want to prove here that equality holds: that $SO(n)$ is *not* simply connected. The idea is to find a finite-dimensional vector space S and a Lie algebra homomorphism

$$\phi\colon \mathfrak{o}(n) \to \mathfrak{gl}(S) \tag{4.1b}$$

which is *not* the differential of a Lie group homomorphism

$$\Phi\colon SO(n) \to GL(S). \tag{4.1c}$$

Here is the strategy. Fix an orthonormal basis $(e_1, \ldots, e_n)$ of $V = \mathbb{R}^n$, and consider the linear map

$$Xe_1 = e_2, \quad Xe_2 = -e_2, \quad Xe_j = 0 \quad (j \geq 3). \tag{4.1d}$$

This matrix is zero except for a two by two block $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the upper left corner. Then $X \in \mathfrak{o}(n)$, and $\exp(2\pi X) = I_V$. So it will be enough to find S and ϕ so that

$$\exp(2\pi\phi(X)) \neq I_S. \tag{4.1e}$$

Here is how we do this. Define

$$S = C(V), \tag{4.1f}$$

a vector space of dimension 2^n. Because $C(V)$ associative, left multiplication provides an algebra homomorphism

$$\delta\colon C(V) \hookrightarrow \mathrm{End}(C(V)). \tag{4.1g}$$

Proposition 3.2 provides a Lie algebra homomorphism

$$\mathfrak{o}(n) \simeq \bigwedge^2 V \hookrightarrow C(V); \tag{4.1h}$$

composing these two maps gives a Lie algebra homomorphism

$$\phi\colon \mathfrak{o}(n) \to \mathfrak{gl}(C(V)). \tag{4.1i}$$

According to (3.3e), the matrix X maps to $-e_1e_2/2 \in C(V)$. Left multiplication by $-e_1e_2$ satisfies

$$\begin{aligned} \phi(X)e_1 &= (-e_1e_2/2)e_1 = e_1^2e_2/2 = -e_2/2, \\ \phi(X)e_2 &= (-e_1e_2/2)e_2 = -e_1e_2^2/2 = e_1/2. \end{aligned} \tag{4.1j}$$

That is, $\psi(X)$ acts on the two-dimensional span of e_1 and e_2 in $C(V)$ by the matrix

$$\phi(X) = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}. \tag{4.1k}$$

Consequently $\exp(2\pi\phi(X))$ acts by $-I$ on the span of e_1 and e_2, proving (4.1e).

This argument is in some sense constructive: it describes the double cover $Spin(n)$ of $SO(n)$ as the group of linear transformations of $C(\mathbb{R}^n)$ (that is, as $2^n \times 2^n$ matrices) having a specified Lie algebra. There is a LOT more to say about these ideas; some of it will appear on the problems for November 9.