Oleg Shamovsky
04/19/05

Maximal parabolic subgroups in $O(V)$

Introduction

The general setting that we are working with in this paper is:

$$V = n\text{-dimensional vector space over a field } F \text{ with char } F \neq 2$$
$$B = \text{non-degenerate orthogonal form on } V$$
$$O(V) = \{g \in GL(V) \mid B(gv, gw) = B(v, w)\}, \text{ where } v, w \in V$$

Definition 1. Let G be a permutation group on a set Ω and x be an element of Ω. Then

$$G_x = \{g \in G \mid g(x) = x\}$$

is called the *stabilizer* of x and consists of all the permutations of G that produce group fixed points in x.

Definition 2. A vector subspace $S \subset V$ is *isotropic* if for any $v, w \in S$, the symmetric bilinear form satisfies:

$$B(v, w) = 0$$

Definition 3. A *maximal parabolic subgroup* in an orthogonal group $O(V)$ is the stabilizer of an isotropic subspace $S \subset V$ in $O(V)$.

We now propose the following variation of Witt's Extension Theorem (proved in the text for quadratic forms on page 41). Suppose that S and S' are k-dimensional isotropic subspaces of the orthogonal vector space V. Then there is an element $g \in O(V)$ such that $g \cdot S = S'$. The proof of this theorem is very similar to the proof given on pages 1-2 of the supplementary notes in "symparabolic.pdf". As a consequence of the above variation of Witt's Extension Theorem, any two subspaces S and S' of the same dimension are conjugate by $O(V)$.

Definition 5. Let S be an isotropic subspace of V and $k \geq 0$. The *isotropic Grassmannian* of V is the collection of all k-dimensional isotropic subspaces of V, namely:

$$IG(k, V) = \{S \subset V \mid \dim S = k\}$$

We know that the maximum possible dimension for an isotropic subspace $S \subset V$ is the Witt index m of V. Therefore, for each $0 \leq k \leq m$, there is an isotropic Grassmannian $IG(k,V)$ consisting of all k-dimensional isotropic subspaces. If S is one such subspace and $P(S)$ is its stabilizer, then we have that:

$$IG(k,V) = O(V)/P(S)$$

Our goal in this paper is to work out the structure of $P(S)$ precisely.

The structure of $P(S)$

In this section, we work out the structure of the stabilizer group $P(S)$ for a k-dimensional isotropic subspace S.

Let $\{e_i\}$ be a basis for $S \simeq F^k$. We wish to find $T \simeq F^k$, an isotropic subspace of basis of V with basis $\{f_j\}$ such that

$$B(e_i, f_j) = 1 \text{ whenever } i = j\text{, and}$$
$$B(e_i, f_j) = 0 \text{ whenever } i \neq j$$

For the purposes of this presentation, I will assume that we have T.

We now define W to be the orthogonal complement of $S \oplus T$ as follows:

$$W = (S \oplus T)^{\perp} = \{w \in V \mid B(w, e_i) = B(w, f_i) = 0 \quad (1 \leq i \leq k)\}$$

Then, by proposition 2.9 of the text, we have that:

$$V = (S \oplus T) \oplus W$$

Then, a typical element $v \in V$ may be written as a triple, as follows:

$$v = (s, t, w)\text{, where } s, t \in F^k\text{, and } w \in W$$

The definition of W and

$$B(e_i, f_j) = 1 \text{ whenever } i = j\text{, and}$$
$$B(e_i, f_j) = 0 \text{ whenever } i \neq j$$

show that the orthogonal form is:

$$B(s_1, t_1, w_1), (s_2, t_2, w_2)) \ = t_2^{tr} s_1 + t_1^{tr} s_2 + B(w_1, w_2)$$

(Note: t^{tr} denotes the transpose of the $k \times 1$ column vector t, so the product $t^{tr}s$ is a scalar)

We will now describe an element of $P(S)$ by saying first what it does to the elements of $S_{\{e_i\}} \simeq F^k$, then to elements of $T_{\{f_j\}} \simeq F^k$ and then to elements of $W = (S \oplus T)^{\perp}$. We will then use the formula of the orthogonal form derived above to test whether the defined elements represent the orthogonal form B.

What we will demonstrate is that any element of $P(S)$ has a unique decomposition as the element of $GL(k)$, element of $O(W)$ and element of $N(S)$, where $O(W)$ is an orthogonal group (acting trivially on S and T) and $GL(k)$ is an $k \times k$ invertible matrix g that

1. Preserves S (acting by the matrix g in the basis $\{e_i\}$)
2. Preserves T (acting by the matrix $(g^{-1})^{tr}$ in the basis $\{f_j\}$), and
3. acts trivially on W.

And, $N(S)$ is a normal subgroup of $P(S)$, consisting of elements which are acting trivially on S and on $S^{\perp}/S = (W \oplus S)/S$. The defined elements preserve the orthogonal form B.

Now, we wish to introduce the groups $GL(k)$, $O(W)$, and $N(S)$, and show that every element of $P(S)$ is
uniquely a product of these groups.

Defining $O(W)$**:**
As an example (from page 4 of the "symparabolic.pdf" notes), suppose $D \in GL(k,F)$ is an invertible $k \times k$ matrix. We define a linear transformation a_D of V by

$$a_D(s,t,w) = (Ds,(D^{-1})^{tr}t,w)$$

Using the orthogonal form, we compute:

$$\begin{aligned} B(a_D(s_1,t_1,w_1),a_D(s_2,t_2,w_2)) &= B((Ds_1,(D^{-1})^{tr}t_1),(Ds_2,(D^{-1})^{tr}t_2,w_2) \\ &= (t_2^{tr}D^{-1})(Ds_1) + (t_1^{tr}D^{-1})(Ds_2) + B(w_1,w_2) \\ &= t_2^{tr}s_1 + t_1^{tr}s_2 + B(w_1,w_2) \\ &= B((s_1,t_1,w_1),(s_2,t_2,w_2)) \end{aligned}$$

Therefore, we see that $a_D \in O(V)$. And, since a_D preserves the subspace S, we have that $a_D \in P(S)$.

Defining $GL(k)$**:**
We begin with a skew-symmetric $k \times k$ matrix E. We then define

$$z_E(s,t,w) = (s+Et,t,w)$$

It is clear that z_E preserves S. To check that z_E is orthogonal, we compute

$$\begin{aligned} B(z_E(s_1,t_1,w_1), z_E(s_2,t_2,w_2)) &= B((s_1+Et_1,t_1,w_1),(s_2+Et_2,t_2,w_2)) \\ &= t_2^{tr}(s_1+Et_1) + t_1^{tr}(s_2+Et_2) + B(w_1,w_2) \\ &= t_2^{tr}s_1 + t_1^{tr}s_2 + B(w_1,w_2) + t_2^{tr}Et_1 - t_1^{tr}Et_2 \end{aligned}$$

Since $E^{tr} = E$, the last two terms in the above expression cancel each other (since they are 1×1). And, the first three terms, $t_2^{tr}s_1 + t_1^{tr}s_2 + B(w_1,w_2)$ are simply $B((s_1,t_1,w_1),(s_2,t_2,w_2))$. This shows that z_E is orthogonal. The collection

$$Z = \{z_E \mid E \text{ skew-symmetric } k \times k\}$$

is a subgroup of $P(S)$, with group law given by addition of skew-symmetric matrices.

We proceed by finding a normal subgroup $N(S)$ of $P(S)$, consisting of elements which are acting trivially on S and on $S^{\perp}/S = (W \oplus S)/S$.

Defining $N(S)$**:**
An element of $N(S)$ depends on two arbitrary choices:

- a linear map $C : W \to S$ and a linear map $Q : T \to S$, given
- in the bases $\{e_i\}$ and $\{f_j\}$ by a skew-symmetric $m \times m$ matrix B

(Note: A bilinear form $\langle\ ,\ \rangle$ on a vector space V is *skew-symmetric* if for all $v \in V$ we have that $\langle v,v\rangle = 0$).

Consider a linear map:

$$A : T \to W$$

Let σ be the corresponding orthogonal transformation in $P(S)$. The first requirement is that

$$\sigma(s) = s, \text{ where } s \in S$$

We would like to define σ such that it sends each element $t \in T$ to $t - A(t)$ in W. But, a subspace $\sigma(T)$ where $\sigma = t - A(t)$ is not isotropic and so we cannot use it as part of an orthogonal transformation.

To fix this, we introduce a correction in S by defining a linear map:

$$Q : T \to S$$

which depends on A. Then, we have:

$$\sigma(t) = t + A(t) + Q(t)$$

Using the identifications that $S \simeq F^k$ and $T \simeq F^k$, the linear map Q can be represented by a $k \times k$ matrix Q. Using the orthogonal form that we defined previously, we see that the requirement that vectors in σ span an isotropic space is:

$$t_2^{tr} Q t_1 + t_1^{tr} Q t_2 = B_W(At_1, At_2), \text{ where } t_1, t_2 \in T$$

We now define Q to be the strictly upper triangular $k \times k$ matrix satisfying

$$t_2^{tr} Q t_1 + t_1^{tr} Q t_2 = B_W(At_1, At_2), \text{ where } t_1, t_2 \in T$$

More specifically, Q_{ij}, the (i,j)-entry of Q is:

$$Q_{ij} = B_W(Af_j, Af_i), \text{ where } 1 \leq i \leq j \leq k$$

Now, we define σ on W in such a way that it is an identity on W. However, W is not orthogonal to $\sigma(T)$. Therefore, we introduce another correction in S by defining a linear map:

$$C : W \longrightarrow S$$

which depends on A. Then we have:

$$\sigma(w) = w + C(w)$$

Again, Using the orthogonal form that we defined previously, we see that the requirement that $\sigma(W)$ be orthogonal to $\sigma(T)$ can be written as:

$$t^{tr} C(w) = B_W(A(t), w), \text{ where } t \in T \text{ and } w \in W$$

We can now define C completely in terms of A. We note that the linear map $A : T \longrightarrow W$ is determined by the k vectors

$$A(e_i) = w_i, \text{ where } w_i \in W$$

And, each of the vectors w_i define a linear function $c_i(w) = B_W(w_i, w)$. Now, consider the i-th such function (ie. $c_i(w)$) to be the i-th coordinate function of $C : W \longrightarrow S$, where $S \simeq F^k$. Therefore, we have:

$$
\begin{aligned}
C(w) &= \begin{bmatrix} c_1(w) \\ \vdots \\ c_k(w) \end{bmatrix} \\
&= \begin{bmatrix} B_W(w_1, w) \\ \vdots \\ B_W(w_k, w) \end{bmatrix} \\
&= \begin{bmatrix} B_W(A(e_1), w) \\ \vdots \\ B_W(A(e_k), w) \end{bmatrix}
\end{aligned}
$$

Finally, having defined $Q : T \to S$ and $C : W \to S$, we define the orthogonal map:

$$\sigma(s, t, w) = (s + C(w) + Q(t), t, w + A(t))$$