**SHADOWS of Lie Theory in the world of matroids**

*(joint w/ T. Braden, J. Eberhardt, E. Kowalke)*

**BIG PICTURE**

**UNDERLYING COMBINATORICS**

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Let $I = \{1, \ldots, h\}$ and $\Lambda_0 \subseteq \mathbb{Z}^I$ be a unimodular sublattice (i.e. $\Lambda_0 = \{ x \in \mathbb{Z}^I | (x, y) \in \mathbb{Z}, \forall y \in \mathbb{Z}^I \}$)

$K = \text{Hom}(\mathbb{Z}^I / \Lambda_0, \mathbb{C}^x) \subseteq (\mathbb{C}^x)^I = \text{Hom}(\mathbb{Z}^I, \mathbb{C}^x)$

(Fix a generic character of $K$: $\alpha \in \text{Hom}(K, \mathbb{C}^x) \cong \mathbb{Z}^I / \Lambda_0$)
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a generic cocharacter of $T := (\mathbb{C}^*)^3 \times \mathbb{C}^* \ni \xi \in \text{Hom}(\mathbb{C}^*; T) \cong \Lambda^*$

Using Hyper-Kähler reduction one gets: "Hypertoric variety" $M_\alpha = \mu_\alpha^{-1}(0)/K \cong \mathbb{C}^*$

The datum $(\Lambda_0, \alpha, \xi)$ also defines a "LINEAR PROGRAM"

- affine hyperplane arrangement:
  let $V = \Lambda_0 \otimes \mathbb{R} + \alpha \subset \mathbb{R}^I$
  and $H_i = V \cap \{ x \in \mathbb{R}^I \mid x_i = 0 \}$

- $\xi$ defines an "objective functional" $\xi \in \Lambda^*$

Thus [Bredon - Licata - Proudfoot - Webster]

Define hypertoric category $\mathcal{O}(\Lambda_0, \alpha, \xi)$

and show $\mathcal{O}(\Lambda_0, \alpha, \xi) \cong A(\Lambda_0, \alpha, \xi) - \text{mod}$

where $A(\Lambda, \alpha, \xi)$ is quasi-hereditary and Koszul.

w/ Koszul dual $A(\Lambda_0^\perp, \xi, \alpha)$

Simple objects in $\mathcal{O}(\Lambda_0, \alpha, \xi)$ are labelled by chambers

Moreover, as $\alpha, \xi$ vary, the $\mathcal{O}(\Lambda_0, \alpha, \xi)$ are derived Morita equivalent.

Thus [Bredon] Fix $\Lambda_0$ as above, $k$ field

$\text{Per}_T(M_\alpha; k) \cong R_k(\Lambda_0) - \text{mod}$

w.r.t. symplectic leaves $k$ is a f.d. alg. "hypertoric Schur alg."
In fact, we can define $R_k(\Lambda_0)$ for any MATROID.

MATROIDS: Suppose $E \subset \mathbb{R}^d$ is a finite spanning set.
Let $B = \{\text{subsets of } E \text{ that form a basis for } \mathbb{R}^d\}$.

\[ \text{Note: (B1) } B \neq \emptyset \]
\[ \text{ (B2) if } X, Y \in B \text{ and } x \in X \setminus Y, \text{ then } \exists y \in Y \setminus X \text{ such that } (X \setminus x) y \in B. \]

Def (Nakahara, Whitney 1935)
A set $E$ with a set of subsets $B$ is a MATROID $M = (E, B)$ if (B1) + (B2) hold.

EXAMPLES: ① LINEAR MATROIDS ($E, B$ as above)
② Graphical matroid: $G$-graph $\rightarrow E = \text{edges of } G$
$B = \text{spanning trees of } G$

(② = ①)

Then [Nelson '18] almost all matroids are not LINEAR!

Can define matroids in terms of their:
- independent sets
- CIRCUITS (minimal dependencies)
- FLATS (spanned sets)

MATROID DUALITY
$M = (E, B)$ matroid $\rightarrow M^* = (E, B^\perp)$, where $B^\perp = \{E \setminus B \mid B \subset B\}$ is also a MATROID (the DUAL matroid)

Note: $\Lambda_0 \subset \mathbb{Z}^I$ is a MATROID (the DUAL matroid)
$E = \{\psi(e_1^*), \ldots, \psi(e_n^*)\} \subset (\Lambda_0 \otimes \mathbb{R})^*$
where $\psi : (\mathbb{R}^I)^* \rightarrow (\Lambda_0 \otimes \mathbb{R})^*$.
We define for any matroid $M$, a MATROIDAL SCHUR alg. $R_k(M)$ s.t. $R_k(M_0) = R_k(M)_{M_0}$

**Question:** Is there a natural q-Sehun algebra?

$R_k(M)$ is defined as the subalgebra of $End(B)$ generated by certain operators and their adjoints.

**Observation:** the vector space $B \cong K$ (hypertoric category $0$) when $M = M_{\Lambda_0}$

Work in progress w/ J. Eberhardt:

Category $R(\Lambda_0)$ using a category of hypertoric Harish-Chandra modules. As $O(\Lambda, \alpha, \delta)$ is Koszul grading of $R_k(\Lambda_0)$.

Q: What about $R(M)$ for $M$ non-linear?

Idea: When $M$ is orientable - can still define a notion of category $0$.

**Oriented Matroids**:

An example: given an oriented graph

\[ \begin{array}{cc}
  a & b \\
  b & c \\
  c & d \\
\end{array} \]

the circuits $\{ bcd, bce, de \}$ can be given orientations.

oriented circuits $\{ bcd, bce, de, 
\quad \overline{bcd}, \overline{bce}, \overline{de} \}$

An oriented matroid can be defined as a set of (oriented) circuits $C \in \{0, +, -\}$.

Topological realization Thm (Fulton + Lawrence):

Oriented matroids can be represented as an arrangement.
of \textbf{PSEUDO SPHERES} \( \{(S_e)_{e \in E}\} \) of \((d-1)\)-spheres \( S_e \) embedded in \( S^d \) such that \( S^d \setminus S_e \cong D^d \cup D^d \) for each \( e \in E \), satisfying 
\[(A_1) \quad S^d_e = \bigcap_{e \in A} S_e \text{ for any } A \subseteq E \text{ is a sphere.} \]

This hints at how one can do (non)linear programming via oriented matroids.

An \textbf{ORIENTED MATROID PROGRAM} is a oriented matroid 
\((\widetilde{M}, g, f)\) w/ two distinguished elements \( g, f \)

\( \text{joint work w/ Ethan Kowalski} \)

\( \text{The } (g, f \text{ sufficiently generic} \) (and a choice of LSOP for the } k[M]\)

we define a f.d. alg. \( A(\widetilde{M}, g, f, U) \)

\( \text{generalizing the alg. of } [BLPW] \)

w/ simple objects are labeled by “f-bounded” chambers (types)

If the program \((\widetilde{M}, f, g)\) is \textbf{EUCLIDIAN,}

Then \( A(\widetilde{M}, f, g, U) \) is quasi-hereditary and Koyznl

\( \text{Rmk: If } (\widetilde{M}, f, g) \text{ is NOT Euclidean, } A(\widetilde{M}, f, g, U) \text{ is NOT } \)

\( \text{don't know about Koyzln.} \)

\( \text{Work in progress: hope to show that as we vary } f, g \)

\( \text{the algebras are derived Morita equivalent} \)