SHADOWS of Lie Theory in the world of matroids

(joint w/ T. Braden, J. Eberhardt, E. Kowalenko)

BIG PICTURE

UNDERLYING COMBINATORICS	GEOMETRY		REP'N THY
root datum	Springer resolution $T^*G/B \simeq \tilde{N} \to N$	$\xleftarrow{\text{LOCALIZATION}}\dashrightarrow$	CATEGORY $\mathcal{O}$
	Nilpotent cone $N \subset gl_n$	$\xleftarrow{\text{Perverse sheaves}}\dashrightarrow$	Schur algebra $S_k(n,n)$
LINEAR PROGRAM (Hyperplane arrang. + functional)	SMOOTH HYPERTORIC VARIETY $\mathcal{M}_\alpha \to \mathcal{M}_0$	[BLPW] $\dashleftarrow\dashrightarrow$	Hypertoric category $\mathcal{O}$
			↕ [EM]
CENTRAL Hyperplane arrangement	Affine hypertoric variety $\mathcal{M}_0$	[BM] $\dashleftarrow\dashrightarrow$	"Hypertoric" Schur algebra
ORIENTED MATROID PROGRAMS	?	$\dashleftarrow\dashrightarrow$	category $\mathcal{O}$ for ORIENTED MATROIDS
			↓
MATROID	?	[BM] $\dashleftarrow\dashrightarrow$	MATROIDAL SCHUR ALGEBRA

STARTING point for [BLPW]

- Let $I = \{1, \dots, n\}$ and $\Lambda_0 \subset \mathbb{Z}^I$ a <u>unimodular</u> sublattice

(i.e. $\Lambda_0 = \{x \in \mathbb{Z}^I \mid \langle x, y\rangle \in \mathbb{Z},\ \forall y \in \Lambda_0\}$)

$\rightsquigarrow$ Torus

$$K := \mathrm{Hom}(\mathbb{Z}^I/\Lambda_0, \mathbb{C}^\times) \subset (\mathbb{C}^\times)^I = \mathrm{Hom}(\mathbb{Z}^I, \mathbb{C}^\times)$$

(Fix a generic character of K : $\alpha \in \mathrm{Hom}(K, \mathbb{C}^\times) \simeq \mathbb{Z}^I/\Lambda_0$

↳ a generic cocharacter of $T := (\mathbb{C}^\times)^I / K$: $\xi \in \mathrm{Hom}(\mathbb{C}^\times, T) \cong \Lambda_0^*$

⇝ using HyperKähler reduction one gets a "Hypertoric variety"

$$\begin{array}{c} \mathfrak{M}_\alpha = \mu^{-1}(0) /\!\!/_{\alpha} K \overset{\xi}{\circlearrowleft} \mathbb{C}^\times \\ \downarrow \\ \mathfrak{M}_0 = \mu^{-1}(0)/\!\!/ K \end{array}$$

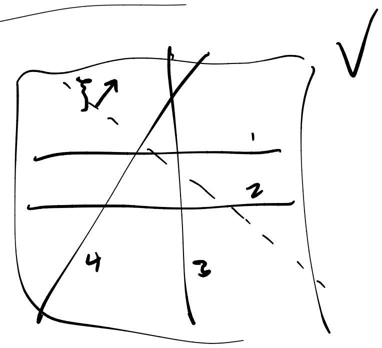
The datum (Λ_0, α, ξ) also defines a "LINEAR PROGRAM"

- affine hyperplane arrangement:

  let $V = \Lambda_0 \otimes \mathbb{R} + \alpha \subset \mathbb{R}^I$

  and $H_i = V \cap \{x \in \mathbb{R}^I \mid x_i = 0\}$

- ξ defines an "objective functional" $\xi \in \Lambda_0^*$



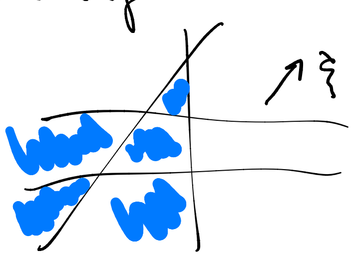
<u>Thm</u> [Braden-Licata-Proudfoot-Webster]

Define hypertoric category $\mathcal{O}(\Lambda_0, \alpha, \xi)$ and show $\mathcal{O}(\Lambda_0,\alpha,\xi) \simeq \underbrace{A(\Lambda_0,\alpha,\xi)}_{\text{f.d. alg.}}$-mod

where $A(\Lambda_0,\alpha,\xi)$ is <u>quasi-hereditary</u> and <u>Koszul</u>.

w/ Koszul dual $A(\underbrace{\Lambda_0^\perp, \xi, \alpha}_{\text{Gale dual}})$

- Simple objects in $\mathcal{O}(\Lambda_0, \alpha, \xi)$ are labelled by chambers bounded by ξ



- Moreover, as α, ξ vary, the $\mathcal{O}(\Lambda_0, \alpha, \xi)$ are derived Morita equivalent.

<u>Thm</u> [Braden] Fix Λ_0 as above, k field

$$\mathrm{Perv}_{T,c}(\mathfrak{M}_0; k) \simeq R_k(\Lambda_0)\text{-mod}$$

c: w.r.t. symplectic leaves

R_k is a f.d. alg "hypertoric Schur alg"

- $R_k(\Lambda_0)$ is quasi-hereditary w/ Ringel dual $R_k(\Lambda_0^\perp)$

In fact, we can define $R_k(\Lambda_0)$ for any MATROID.

MATROIDS: Suppose $E \subset \mathbb{R}^d$ is a finite spanning set.

Let $\mathcal{B} = \{$subsets of E that form a basis for $\mathbb{R}^d\}$

[Note: (B1) $\mathcal{B} \neq \emptyset$

(B2) If $X, Y \in \mathcal{B}$ and $x \in X \setminus Y$, then

$\exists\, y \in Y \setminus X$ such that $(X \setminus x) \cup y \in \mathcal{B}$]

Def (Nakasawa, Whitney 1935)

A set E w/ a set of subsets $\mathcal{B}$ is a MATROID $M = (E, \mathcal{B})$ is (B1) + (B2) hold.

EXAMPLES: ① LINEAR MATROIDS (E, $\mathcal{B}$ as above)

② Graphical matroid: G-graph $\rightsquigarrow$ E = edges of G

$\mathcal{B}$ = spanning trees of G

(② ⊂ ①)

Thm [Nelson '18] almost all matroids are not LINEAR!

[Can define matroids in terms of their:
- independent sets
- CIRCUITS (minimal dependencies)
- FLATS (spanned sets)]

MATROID DUALITY

$M = (E, \mathcal{B})$ matroid $\rightsquigarrow$ $M^* = (E, \mathcal{B}^\perp)$, where $\mathcal{B}^\perp = \{E \setminus B \mid B \in \mathcal{B}\}$

is also a MATROID (the DUAL MATROID)

Note: $\Lambda_0 \subset \mathbb{Z}^I \rightsquigarrow$ MATROID

$E = \{\varphi(e_1^*), \ldots, \varphi(e_n^*)\} \subset (\Lambda_0 \otimes \mathbb{R})^*$

where $\varphi : (\mathbb{R}^I)^* \longrightarrow (\Lambda_0 \otimes \mathbb{R})^*$.

[Braden-M] We define for any matroid M,
a MATROIDAL SCHUR alg. $R_k(M)$ s.t. $R_k(\Lambda_0) = R_k(M_{\Lambda_0})$

<u>Question</u>: Is there a natural q-Schur algebra?

$R_k(M)$ is defined as the subalg of $\mathrm{End}(\mathbb{B})$ ← large vector space
generated by certain operators and their adjoints.

<u>OBSERVATION</u>: the vector space $\mathbb{B} \cong K$(hypertoric category $\mathcal{O}$)
when $M = M_{\Lambda_0}$

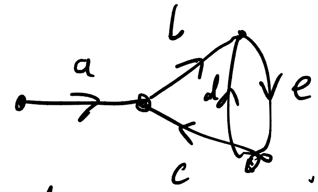
<u>Work in progress w/ J. Eberhardt</u>:

Categorify $R(\Lambda_0)$ using a category of hypertoric Harish-Chandra bimodules. As $\mathcal{O}(\Lambda_0, \alpha, \xi)$ is Koszul ⇝ grading ⇝ q-version of $R_k(\Lambda_0)$.

<u>Q</u>: What about $R(M)$ for M non-linear?

<u>Idea</u>: When M is <u>orientable</u> - can still define a notion of category $\mathcal{O}$.

<u>ORIENTED MATROIDS</u>:

<u>An example</u>: given an oriented graph 

the circuits $\{bcd, bce, de\}$ can be given orientations
⇝ oriented circuits $\{bc\bar{d}, bce, de,$
$\bar{b}\bar{c}d, \bar{b}\bar{c}\bar{e}, \bar{d}\bar{e}\}$

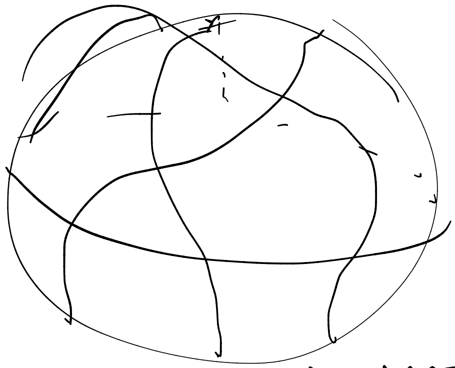
An <u>oriented matroid</u> can be defined as a set of (oriented) circuits
$C \in \{0, +, -\}^I$.

<u>Topolog. realization Thm</u> (Folkman + Lawrence)
oriented matroids can be represented as an arrangement

of PSEUDO SPHERES [a collection $(S_e)_{e\in E}$ of $(d-1)$ spheres S_e embedded in S^d such that $S^d \setminus S_e \underset{\text{homeo.}}{\simeq} D^d_+ \sqcup D^d_-$ $\forall e \in E$.

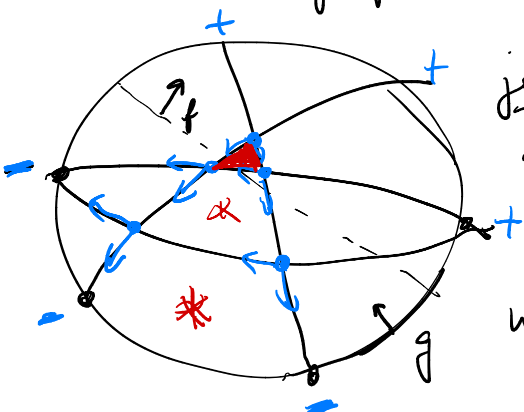
satisfying : (A1) $S_A = \bigcap_{e\in A} S_e$ for any $A \subset E$ is a sphere.

...



This hints at how one can do (non)linear programming via oriented matroids.

An ORIENTED MATROID PROGRAM is a oriented matroid $(\tilde{M}, g, f)$ w/ two distinguished elements g, f



joint work w/ Ethan Kowalenko

Thm For g, f sufficiently generic (and a choice U of L.S.O.P. for the $k[M^{un}]$) we define a f.d. alg. $A(\tilde{M}, g, f, U)$ generalizing the alg. of [BLPW] w/ simple objects are labelled by "f-bounded" chambers (topes)

If the program $(\tilde{M}, f, g)$ is EUCLIDEAN, Then $A(\tilde{M}, f, g, U)$ is quasi-hereditary and Koszul

Rmk: If $(\tilde{M}, f, g)$ is NOT Euclidean, $A(\tilde{M}, f, g, U)$ is NOT q.hered. Don't know about Koszul.

Work in progress : hope to show that as we vary f, g the algebras are derived Morita equivalent.