On the unitary dual of classical and exceptional real split groups.

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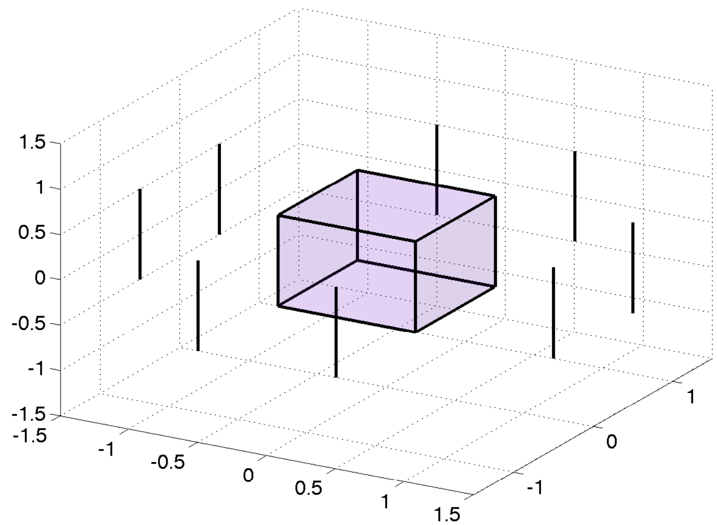


PLAN OF THE TALK

petite K-types

$$\Downarrow$$

unitarizability of Langlands quotients for real split groups	$\overset{?}{\Longleftrightarrow}$	unitarizability of Langlands quotients for Hecke algebras

- *Part 1.* Spherical unitary dual of affine graded Hecke algebras
- *Part 2.* Unitary dual of real split groups
- *Part 3.* Petite K-types
- *Part 4.* Embedding of unitary duals

PART 1

Spherical unitary dual of affine graded Hecke algebras

Affine graded Hecke algebras

- $(X, \Delta, \check{X}, \check{\Delta})$: root datum, with simple roots Π and Weyl group W
- $\mathfrak{h} := X \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathbb{A} := Sym(\mathfrak{h})$.

The **affine graded Hecke algebra** for Δ is $\boxed{\mathbb{H} := \mathbb{C}[W] \otimes \mathbb{A}}$.

generators: $\{t_{s_\alpha} : \alpha \in \Pi\} \cup \{\omega : \omega \in \mathfrak{h}\}$

relations: $t_{s_\alpha}^2 = 1$; $\omega t_{s_\alpha} = t_{s_\alpha} s_\alpha(\omega) + \langle \omega, \check{\alpha} \rangle \quad \alpha \in \Pi, \omega \in \mathfrak{h}$

Example: type $A1$.

- ◇ $\mathbb{C}[W] = \mathbb{C}1 + \mathbb{C}t_{s_\alpha}$ $(t_{s_\alpha}^2 = 1)$, $\mathbb{A} = Sym(\mathbb{C}\alpha)$
- ◇ $\alpha t_{s_\alpha} = -t_{s_\alpha}\alpha + 2$.
- ◇ Elements of $\mathbb{H} := \mathbb{C}[W] \otimes \mathbb{A}$ are of the form: $p(\alpha) + q(\alpha)t_{s_\alpha}$.

Spherical Unitary $\mathbb{H}$-modules

The Hecke algebra $\mathbb{H} := \mathbb{C}[W] \otimes \mathbb{A}$ has a $\star$-operation.

Let V be an $\mathbb{H}$-module.

- V is **spherical** if it contains the trivial representation of W.
- V is **Hermitian** if it has a non-degenerate invariant Hermitian form:

$$\langle x \cdot v_1, v_2 \rangle + \langle v_1, x^\star \cdot v_2 \rangle = 0 \qquad x \in \mathbb{H},\ v_1, v_2 \in V.$$

- V is **unitary** if $\langle , \rangle$ is *positive definite.*

Spherical Langlands quotients of $\mathbb{H}$

- $\diamond$ $\nu \in \check{\mathfrak{h}} \simeq \mathfrak{h}^*$: real and dominant
- $\diamond$ $\mathbb{C}_\nu$: the corresponding character of $\mathbb{A} = Sym(\mathfrak{h})$

Spherical Principal Series $\boxed{X(\nu) := \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu}$ ($\mathbb{H}$ acts on the left)

$X(\nu)$ is spherical, $\dim \mathrm{Hom}_W(triv, X(\nu)) = 1$.

Langlands Quotient $\boxed{L(\nu) := \text{unique irreducible quotient of } X(\nu)}$

Every irreducible spherical $\mathbb{H}$-module (with real central character) is isomorphic to $L(\nu)$ for some ν dominant.

An Hermitian form on $L(\nu)$

- ◇ w_0 : longest Weyl group element
- ◇ $L(\nu)$ is Hermitian if and only if $w_0\nu = -\nu$.

In this case, Barbasch and Moy define a (normalized) intertwining operator

$$A(w_0, \nu)\colon X(\nu) \to X(w_0\nu)$$

such that

- $A(w_0, \nu)$ has no poles
- The image of $A(w_0, \nu)$ is $L(\nu)$, hence $L(\nu) \simeq \frac{X(\nu)}{Ker(A(w_0,\nu))}$
- $A(w_0, \nu)$ gives a non-deg. invariant Hermitian form on $L(\nu)$.

$\forall\, \psi \in \widehat{W}$, we find an operator $A_\psi(w_0, \nu)$ on V_ψ^*. $(A_{triv}(w_0, \nu) = 1.)$

The operator $A_\psi(w_0,\nu)$ on the W-type ψ

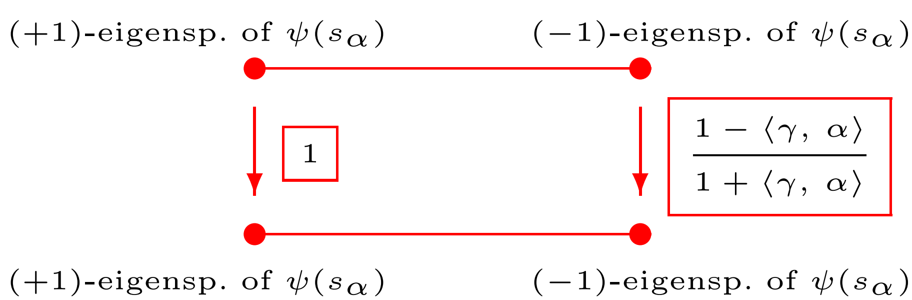
$A_\psi(w_0,\nu)$ decomposes a product of operators corresponding to simple reflections.

If $w_0 = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$ is a reduced decomposition of w_0 in W, then

$$A_\psi(w_0,\nu) = \prod_{i=1}^{n} A_\psi(s_{\alpha_i}, \nu_{i-1})$$

with $\nu_i = s_{\alpha_{n+1-i}} \cdots s_{\alpha_{n-1}} s_{\alpha_n} \nu$.
Each factor acts on V_ψ^*, and has the form

$$\boxed{A_\psi(s_\alpha,\gamma) := \frac{Id + \langle\alpha,\gamma\rangle \psi(s_\alpha)}{1+\langle\alpha,\gamma\rangle}}.$$



Untarity of a spherical Langlands Quotient

$L(\nu)$ is unitary $\Leftrightarrow$ $A_\psi(w_0, \nu)$ is positive semi-definite, $\forall\, \psi \in \widehat{W}$

Barbasch and Ciubotaru found a small subset of $\widehat{W}$ which is sufficient to detect unitarity. These are the ***relevant W-types***.

A_{n-1}	$\{(n-m, m) \colon 0 \leq m \leq [n/2]\}$
B_n, C_n	$\{(n-m, m) \times (0) \colon 0 \leq m \leq [n/2]\}$ $\cup\{(n-m) \times (m) \colon 0 \leq m \leq n\}$
D_n	$\{(n-m, m) \times (0) \colon 0 \leq m \leq [n/2]\}$ $\cup\{(n-m) \times (m) \colon 0 \leq m \leq [n/2]\}$
G_2	$\{1_1, 2_1, 2_2\}$
F_4	$\{1_1, 4_2, 2_3, 8_1, 9_1\}$
E_6	$\{1_p, 6_p, 20_p, 30_p, 15_q\}$
E_7	$\{1_a, 7'_a, 27_a, 56'_a, 21'_b, 35_b, 105_b\}$
E_8	$\{1_x, 8_z, 35_x, 50_x, 84_x, 112_z, 400_z, 300_x, 210_x\}$

Spherical Unitary dual of a Hecke algebra of type $A1$

In type $A1$, there are two W-types ($triv$ and sgn).

$L(\nu)$ is unitary if and only if $A_\psi(w_0, \nu)$ is positive semidefinite for $\psi = triv$ and sgn.

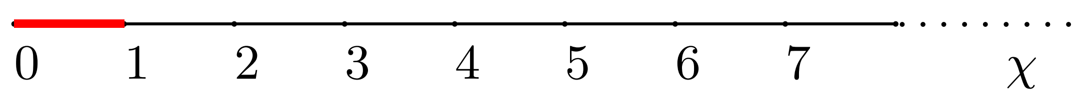
Because w_0 has length one, the intertwining operator has a single factor:

$$A_\psi(w_0, \nu) = A_\psi(s_\alpha, \nu) := \frac{Id + \langle \alpha, \nu \rangle \psi(s_\alpha)}{1 + \langle \alpha, \nu \rangle}.$$

ψ	$\psi(s_\alpha)$	$A_\psi(w_0, \nu)$
$triv$	1	1
sgn	-1	$\frac{1-\nu}{1+\nu}$

$\Rightarrow$ $L(\nu)$ is unitary for $0 \leq \nu \leq 1$.

$\mathbb{H}(A1)$



Spherical Unitary dual of a Hecke algebra of type $B2$

Here $w_0 = s_{e_1-e_2}s_{e_2}s_{e_1-e_2}s_{e_2}$. For $\nu = (a, b)$, $A_\psi(w_0, \nu)$ factors as:

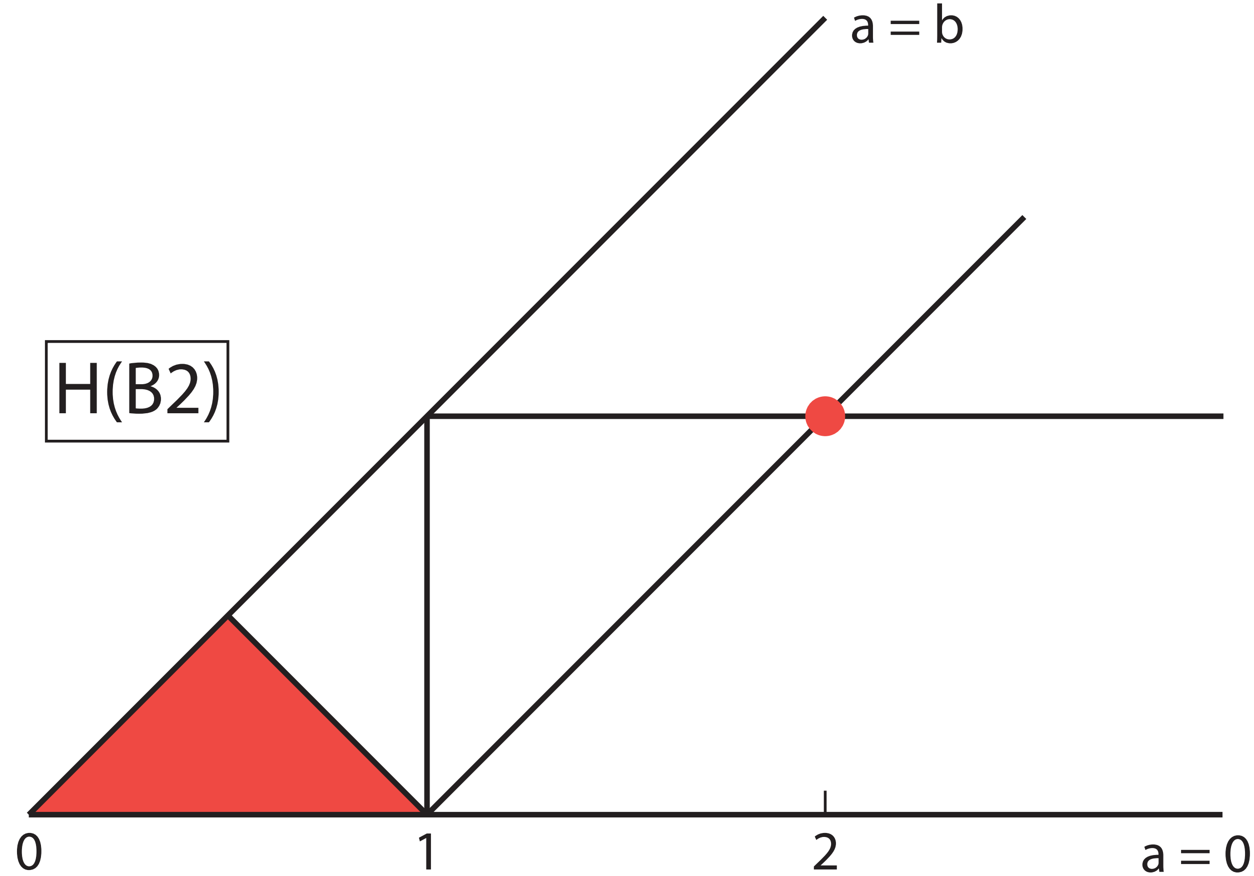
$$A_\psi(s_{e_1-e_2}, (-b, -a))A_\psi(s_{e_2}, (-b, a))A_\psi(s_{e_1-e_2}, (a, -b))A_\psi(s_{e_2}, (a, b))$$

The factors are computed using the formula:

$$A_\psi(s_\alpha, \gamma) = \frac{Id+\langle\alpha,\gamma\rangle\psi(s_\alpha)}{1+\langle\alpha,\gamma\rangle}.$$

$\psi \in \widehat{W}_{rel}$	$\psi(s_{e_1-e_2})$	$\psi(s_{e_2})$	the operator $A_\psi(w_0, \nu)$
2×0	1	1	$1 \cdot 1 \cdot 1 \cdot 1$
11×0	-1	1	$\frac{1-(a-b)}{1+(a-b)} \cdot 1 \cdot \frac{1-(a+b)}{1+(a+b)} \cdot 1$
0×2	1	-1	$1 \cdot \frac{1-a}{1+a} \cdot 1 \cdot \frac{1-b}{1+b}$
1×1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$tr.\quad 2\frac{1+a^2-a^3b-b^2+ab+ab^3}{(1+a)(1+b)[1+(a-b)][1+(a+b)]}$ $det\quad \frac{1-a}{1+a}\frac{1-b}{1+b}\frac{1-(a-b)}{1+(a-b)}\frac{1-(a+b)}{1+(a+b)}$

Spherical unitary dual of a Hecke algebra of type $B2$



The spherical unitary dual of a Hecke algebra

◇ $\mathbb{H}$: affine graded Hecke algebra with root system Δ

◇ $\mathfrak{g}$: complex Lie algebra associated to Δ

The spherical unitary dual of $\mathbb{H}$ is a union of sets (*complementary series*) parameterized by nilpotent orbits in $\mathfrak{g}$.

Fix an orbit $\mathcal{O}$ and its Lie triple $\{e, h, f\}$. The centralizer in $\mathfrak{g}$ of the Lie triple is a reductive Lie subalgebra, denoted $\mathcal{Z}_{\mathfrak{g}}(\mathcal{O})$.
The complementary series attached to $\mathcal{O}$, denoted by $CS(\mathcal{O})$, is the set of all parameters ν such that

- $L(\nu)$ is unitary
- $\mathcal{O}$ is the maximal nilpotent orbit with the property

$$\nu = \tfrac{1}{2}h + \chi, \text{ with } \chi \in \mathcal{Z}_{\mathfrak{g}}(\mathcal{O}).$$

Complementary series attached to nilpotent orbits

Fix a Hecke algebra $\mathbb{H}$ and an orbit $\mathcal{O}$. Barbasch and Ciubotaru reduce the problem of finding the complementary series $CS(\mathcal{O})$ of $\mathbb{H}$ to the problem of detecting the *zero*-complementary series $CS(0)$ of the centralizer of $\mathcal{O}$.

With a few exceptions (one orbit for $F4$ and E_7, and 6 orbits for $E8$), they prove that

$$\nu = \frac{1}{2}h + \chi \in CS_{\mathbb{H}}(\mathcal{O}) \Leftrightarrow \chi \in CS_{\mathbb{H}(\mathcal{Z}(\mathcal{O}))}(0).$$

The *zero*-complementary series

$$CS(0) = \{\nu \colon X(\nu) \text{ is unitary and irreducible}\}$$

is known explicitly (*for all root systems*). It is a union of alcoves in the complement of the reduciblity hyperplanes: $\langle \alpha, \nu \rangle = 1$, $\alpha \in \Delta^+$.

Spherical unitary dual of a Hecke algebra of type $B2$

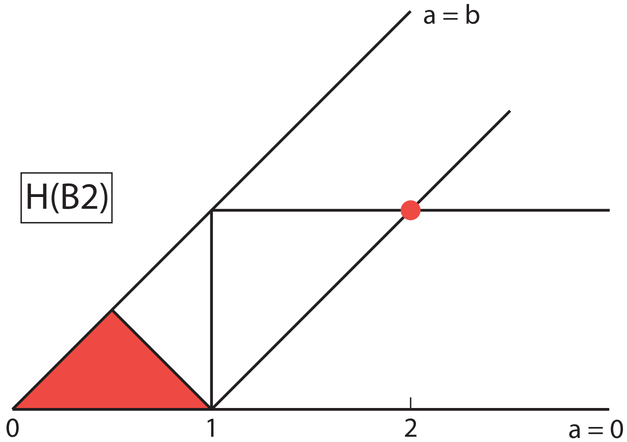
Complementary series are attached to nilpotent orbits in $\mathfrak{so}(5,\mathbb{C})$.

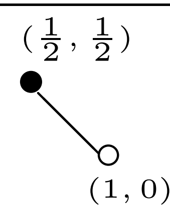
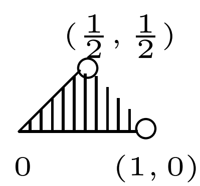
$$\nu = \frac{1}{2}h + \chi \in CS_{\mathbb{H}}(\mathcal{O}) \Leftrightarrow \chi \in CS_{\mathbb{H}(\mathcal{Z}(\mathcal{O}))}(0).$$

The zero-complementary series for the centralizers are known.

$\mathcal{O}$	$\frac{1}{2}h$	$\chi \in \mathcal{Z}(\mathcal{O})$	$\frac{1}{2}h + \chi \in CS_{\mathbb{H}}(\mathcal{O})$
5	$(1,2)$	$(0,0)$	$(1,2)$
311	$(0,1)$	$(a,0)$	$(a,1)$ $\quad a=0$
221	$(-\frac{1}{2},\frac{1}{2})$	(a,a)	$(-\frac{1}{2}+a,\frac{1}{2}+a)$ $0 \leq a < \frac{1}{2}$
11111	$(0,0)$	(a,b)	(a,b) $0 \leq a \leq b < 1-a < 1$

Spherical unitary dual of a Hecke algebra of type $B2$



$(2,1)$ ●	$CS(5)$
$(1,0)$ ●	$CS(3,1,1)$
 $(\frac{1}{2},\frac{1}{2})$, $(1,0)$	$CS(2,2,1)$
 $(\frac{1}{2},\frac{1}{2})$, 0, $(1,0)$	$CS(1,1,1,1,1)$

PART 2

Unitary dual of (double cover of) real split groups

Real split groups

Δ	G	$K \subset G$ (maximal compact)
A_n	$SL(n+1, \mathbb{R})$	$SO(n+1)$
B_n	$SO(n+1, n)_0$	$SO(n+1) \times SO(n)$
C_n	$Sp(2n, \mathbb{R})$	$U(n)$
D_n	$SO(n, n)_0$	$SO(n) \times SO(n)$
G_2	G_2	$SU(2) \times SU(2)/\{\pm I\}$
F_4	F_4	$Sp(1) \times Sp(3)/\{\pm I\}$
E_6	E_6	$Sp(4)/\{\pm I\}$
E_7	E_7	$SU(8)/\{\pm I\}$
E_8	E_8	$Spin(16)/\{I, w\}$

Fine K-types

For each root α we choose a Lie algebra homomorphism

$$\phi_\alpha \colon \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{g}_0$$

whose image is a subalgebra of $\mathfrak{g}_0$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Then $Z_\alpha := \phi_\alpha\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a generator for an $\mathfrak{so}(2)$-subalgebra of $\mathfrak{k}_0$.

A K-type $\mu \in \widehat{K}$ is level k if $\|\gamma\| \le k$ for every root α and every eigenvalue γ of $\mathrm{d}\mu(Z_\alpha)$.

K-types of level $\le \frac{1}{2}$ are called "pseudospherical"; the ones of level ≤ 1 are "fine".

Every $\delta \in \widehat{M}$ is contained in a fine K-type μ_δ (maybe not unique). Every M-type in the W-orbit of δ appears in μ_δ with multiplicity 1.

Minimal Principal Series

- **Parameters** $\begin{cases} P = MAN & \text{minimal parabolic subgroup} \\ (\delta, V^\delta) \in \widehat{M} & \\ \nu \in \widehat{A} \simeq \mathfrak{a}^* & \textit{real and weakly dominant} \end{cases}$

- **Principal Series** $\boxed{X(\delta,\nu)} = \boxed{\mathrm{Ind}_{P=MAN}^{G}(\delta \otimes \nu \otimes triv)}$

$$\forall\, \mu \in \widehat{K},\ \mathrm{mult}(\mu, X(\delta,\nu)|_K) = \mathrm{mult}(\delta, \mu|_M).$$

G	$P = MAN$	$M \simeq \mathbb{Z}_2$	$A \simeq \mathbb{R}_{>0}$	δ	ν
$SL_2(\mathbb{R})$	$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$	$\pm I$	$\begin{pmatrix} \lvert a\rvert & 0 \\ 0 & \lvert a\rvert^{-1} \end{pmatrix}$	$triv$ sgn	$\nu \geq 0$

$$\widehat{K} \simeq \mathbb{Z},\ n|_M = \begin{cases} triv & \textit{if } n \in 2\mathbb{Z} \\ sgn & \textit{if } n \in 2\mathbb{Z}+1 \end{cases},\ X(\delta,\nu)|_{SO(2)} = \begin{cases} \bigoplus_n 2n & \textit{if } \delta = triv \\ \bigoplus_n 2n+1 & \textit{if } \delta = sgn \end{cases}$$

The Langlands quotient $L(\delta, \nu)$

For simplicity, assume, that δ is contained in a unique fine K-type μ_δ.

$L(\delta, \nu)$=unique irreducible quotient of $X(\delta, \nu)$ containing μ_δ

If $w_0 \in W$ is the longest element, there is an intertwining operator

$$A(w_0, \delta, \nu) \colon X(\delta, \nu) \to X(w_0\delta, w_0\nu)$$

(normalized on μ_δ) such that

- $A(w_0, \delta, \nu)$ has no poles
- The (closure of the) image of $A(w_0, \delta, \nu))$ is $L(\delta, \nu)$. Hence
$$L(\delta, \nu) = \frac{X(\delta,\nu)}{Ker(A(w_0,\delta,\nu))}.$$

Unitarity of the Langlands quotient $L(\delta,\nu)$

$L(\delta,\nu)$ is Hermitian if and only if

$$w_0\delta \simeq \delta \text{ and } w_0\nu = -\nu.$$

In this case, the (normalized) operator

$$\boxed{\mathcal{A}(w_0,\delta,\nu) := \mu_\delta(w_0)A(w_0,\delta,\nu)\colon X(\delta,\nu) \to X(\delta,-\nu)}$$

induces a *non-degenerate* invariant Hermitian form on $L(\delta,\nu)$.

For every $\mu \in \widehat{K}$, there is an operator $\mathcal{A}_\mu(w_0,\delta,\nu)$ on $\mathrm{Hom}_M(\mu,\delta)$.
$(\mathcal{A}_{\mu_\delta}(w_0,\delta,\nu) = 1.)$

$L(\delta,\nu)$ is unitary if and only if the operator
$\mathcal{A}_\mu(w_0,\delta,\nu)$ is positive semidefinite, $\forall \mu \in \widehat{K}$

$\Rightarrow$ *We should compute the signature of infinitely many operators.*

The operator $\mathcal{A}_\mu(w_0, \delta, \nu)$ on the K-type μ

Fix a K-type μ. The operator $\mathcal{A}_\mu(w_0, \delta, \nu)$ can be written as a product of operators corresponding to *simple* reflections.

If $w_0 = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_1}$ is a reduced decomposition of w_0 in W, then

$$\mathcal{A}_\mu(w_0, \delta, \nu) = \prod_{i=1}^{n} \mathcal{A}_\mu(s_{\alpha_i}, \delta_{i-1}, \nu_{i-1})$$

with $\nu_i = s_{\alpha_{n+1-i}} \cdots s_{\alpha_{n-1}} s_{\alpha_n} \nu$, and $\delta_i = s_{\alpha_{n+1-i}} \cdots s_{\alpha_{n-1}} s_{\alpha_n} \delta$.

Note that the full operator $\mathcal{A}_\mu(w_0, \delta, \nu)$ acts on $\mathrm{Hom}_M(\mu, \delta)$, but the factors might not.

[The reflections move δ around in its W-orbit.]

The "α-factor" $\mathcal{A}_\mu(s_\alpha, \rho, \gamma)$

An "α-factor" of the full intertwining operator is a map

$$\mathcal{A}_\mu(s_\alpha, \rho, \gamma) \colon \mathrm{Hom}_M(\mu, \rho) \to \mathrm{Hom}_M(\mu, s_\alpha \rho).$$

Here ρ and $s_\alpha\rho$ are two M-types in the W-orbit of δ. They can both be realized inside μ_δ (our fixed fine K-type containing δ).

Let Z_α be a generator for the $\mathfrak{so}(2)$-subalgebra attached to α. Then Z_α^2 acts on $\mathrm{Hom}_M(\mu, \rho)$; decompose $\mathrm{Hom}_M(\mu, \rho)$ in Z_α^2-eigenspaces:

$$\mathrm{Hom}_M(\mu, \rho) = \bigoplus_l E^{\alpha,\rho}(-l^2)$$

[Here $l \in \mathbb{Z} + \frac{1}{2}$ if δ is genuine and α is metaplectic; $l \in \mathbb{Z}$ otherwise.]

The operator $\mathcal{A}_\mu(s_\alpha, \rho, \gamma)$ maps $E^{\alpha,\rho}(-l^2) \to E^{\alpha, s_\alpha\rho}(-l^2)$, $\forall\, l$, acting by: $\boxed{T \mapsto c_l(\alpha, \gamma)\mu_\delta(\sigma_\alpha) \circ T \circ \mu(\sigma_\alpha^{-1})}$.

The scalars $c_l(\alpha,\gamma)$

Let $l \in \frac{1}{2}\mathbb{Z}$, $l \geq 0$. Set $\xi = \langle \gamma, \check{\alpha} \rangle$. Then $c_l(\alpha,\gamma)$ is equal to

- $\boxed{1}$ if $l = 0, 1$ or $\frac{1}{2}$ (for the normalization)
- $\boxed{(-1)^{l/2} \dfrac{(1-\xi)(3-\xi)\cdots(2m-1-\xi)}{(1+\xi)(3+\xi)\cdots(l-1+\xi)}}$ if $l \in 2\mathbb{N}$
- $\boxed{(-1)^{(l-1)/2} \dfrac{(2-\xi)(4-\xi)\cdots(l-1-\xi)}{(2+\xi)(4+\xi)\cdots(l-1+\xi)}}$ if $l \in 2\mathbb{N}+1$
- $\boxed{(-1)^{(l+1/2)/2} \dfrac{(\frac{1}{2}-\xi)(\frac{5}{2}-\xi)\cdots(l-1-\xi)}{(\frac{1}{2}+\xi)(\frac{3}{2}+\xi)\cdots(l-1+\xi)}}$ if $l \in \frac{3}{2}+2\mathbb{N}$
- $\boxed{(-1)^{(l-1)/2} \dfrac{(\frac{3}{2}-\xi)(\frac{7}{2}-\xi)\cdots(l-1-\xi)}{(\frac{3}{2}+\xi)(\frac{7}{2}+\xi)\cdots(l-1+\xi)}}$ if $l \in \frac{5}{2}+2\mathbb{N}$

PART 3

Petite K-types

The idea of petite K-types

The unitarity of $L(\delta, \nu)$ is hard to detect:

- ⋄ It depends on the signature of *infinitely many* operators (one for each K-type)
- ⋄ The computations are hard if the K-type is "large".

To make the problem easier:

1. Select a small set of "petite" K-types on which the computations are easy.
2. Compute operators *only* for petite K-types, hoping that the calculation will rule out large non-unitarity regions.

This approach will provide necessary conditions for unitarity:

$$L(\delta, \nu) \text{ unitary } \Rightarrow \mathcal{A}_\mu(w_0, \delta, \nu) \text{ pos. semidefinite, } \forall\, \mu \text{ petite}$$

Main feature of petite K-types

The operators $\{\mathcal{A}_\mu(\delta, \nu) \colon \mu \; \textit{petite}\}$ resemble operators for affine graded Hecke algebras.

unitarizability of Langlands quotients for real split groups	$\overset{relate}{\Longleftrightarrow}$	unitarizability of Langlands quotients for Hecke algebras

Petite K-types

For each root α and each M-type ρ, there is an action of Z_α^2 on the space $\mathrm{Hom}_M(\mu, \rho)$, defined by

$$T \to T \circ \mathrm{d}\mu(Z_\alpha)^2.$$

If μ_δ is a fine K-type containing δ, we look at the action of Z_α^2 on the space

$$\mathrm{Hom}_M(\mu, \mu_\delta) = \bigoplus_{\delta_i \in W\text{-orbit of } \delta} \mathrm{Hom}_M(\mu, \delta_i).$$

Definition. *A K-type is called "petite for δ" if the eigenvalues of Z_α^2 on $\mathrm{Hom}_M(\mu, \mu_\delta)$ are of the form $-k^2$ with $|k| \leq 2$, $\forall\, \alpha \in \Delta$.*

This is a restriction on the eigenvalues of Z_α^2 on the isotypic component in μ of every M-type in the W-orbit of δ.

Corollary. *K-types of level ≤ 2 are petite for every δ.*

Main theorem

- $\diamond$ G: real split group associated to a root system Δ
- $\diamond$ δ: an M-type for G, with fine K-type μ_δ
- $\diamond$ W^δ: stabilizer of δ in W; Δ_δ : good roots for δ
- $\diamond$ $\mathbb{H}$: affine graded Hecke algebra for $\begin{cases} \Delta_\delta & \text{if } \delta \text{ is genuine} \\ \check{\Delta}_\delta & \text{otherwise.} \end{cases}$

Theorem. For each K-type μ, there is a representation ψ_μ of the stabilizer W^δ on the space $\mathrm{Hom}_M(\mu, \delta)$. When μ is *petite*, the correspondence

$$\mu \in \widehat{K} \Longrightarrow \psi_\mu \in \widehat{W^\delta}$$

gives rise to a matching of intertwining operators:

$$\boxed{\mathcal{A}^G_\mu(w_0, \delta, \nu) = A^{\mathbb{H}}_{\psi_\mu}(w_0, \tilde{\nu})}.$$

The spherical case: $\delta = \delta_0$ is trivial

If $\delta = \delta_0$ is trivial, every reflection stabilizes δ. So

- $W^{\delta_0} = W$
- $\Delta_{\delta_0} = \Delta$
- $\boxed{\mathbb{H}} = \boxed{\text{Hecke algebra with root system } \check{\Delta}} \leftarrow$ *δ_0 non-genuine take $\check{\Delta}_{\delta_0}$*

Theorem. If μ is petite for δ_0 and ψ_μ is the representation of W on $\mathrm{Hom}_M(\mu, \delta_0)$, then

$$\boxed{\mathcal{A}^G_\mu(w_0, \delta_0, \nu) \equiv A^{\mathbb{H}}_{\psi_\mu}(w_0, \tilde{\nu})}$$

The spherical case: $\delta = \delta_0$ is trivial

- Decompose $\mathcal{A}_\mu(w_0, \delta, \nu)$ in "α-factors"

$$\mathcal{A}_\mu(s_\alpha, \rho, \lambda) \colon \mathrm{Hom}_M(\mu, \rho) \to \mathrm{Hom}_M(\mu, s_\alpha \rho)$$

(with ρ an M-type in the W-orbit of δ).

- If $\delta = \delta_0$, then $\rho = s_\alpha \rho = \delta_0$, so every factor

$$\mathcal{A}_\mu(s_\alpha, \rho, \lambda) \in \mathrm{End}(\mathrm{Hom}_M(\mu, \delta_0))$$

- $\mathrm{Hom}_M(\mu, \delta_0)$ carries a repr. ψ_μ of W, and an action of $d\mu(Z_\alpha^2)$.
- Decompose $\mathrm{Hom}_M(\mu, \delta_0)$ in eigenspaces of $d\mu(Z_\alpha^2)$:

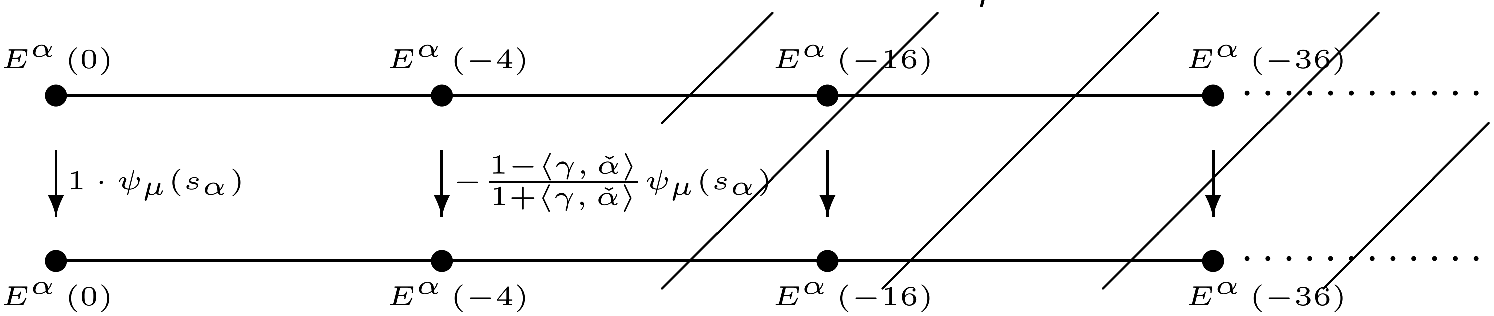
$$\mathrm{Hom}_M(\mu, \delta_0) = \bigoplus_{m \in \mathbb{N}} E^\alpha(-4m^2).$$

- $\mathcal{A}_\mu(s_\alpha, \rho, \lambda)$ acts on acts on $E^\alpha(-4m^2)$ by

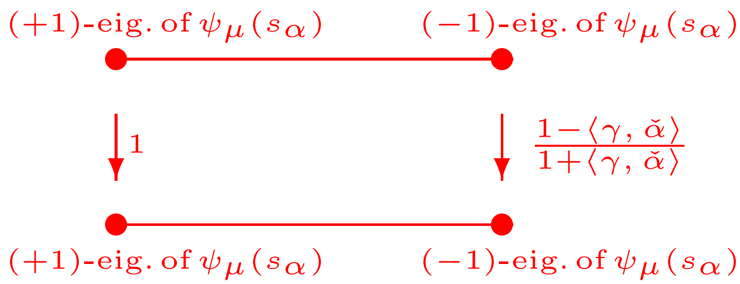
$$T \mapsto \underbrace{c(\alpha, \gamma, 2m)}_{a\ scalar} \underbrace{\mu_\delta(\sigma_\alpha) T (\mu(\sigma_\alpha)^{-1})}_{\psi_\mu(s_\alpha)T}.$$

Petite Spherical K-types

If the K-type μ is petite for δ_0, then $(V_\mu^*)^M = E^\alpha(0) \oplus E^\alpha(-4)$.



The spaces $E^\alpha(0)$ and $E^\alpha(-4)$ coincide with the $(+1)$- and (-1)-eigenspace of $\psi_\mu(s_\alpha)$, respectively.



Hence $\mathcal{A}_\mu(s_\alpha, \delta_0, \lambda) = \frac{I+\langle\gamma,\check{\alpha}\rangle\psi_\mu(s_\alpha)}{1+\langle\gamma,\check{\alpha}\rangle}$. $\Longleftarrow$ *this is an operator for a Hecke algebra with root system $\check{\Delta}$*

The pseudospherical case

If δ is pseudospherical, every reflection stabilizes δ. So

- $\diamond$ $W^\delta = W$
- $\diamond$ $\Delta_\delta = \Delta$
- $\diamond$ $\boxed{\mathbb{H}} = \boxed{\text{Hecke algebra with root system } \Delta} \leftarrow$ *δ genuine take* Δ_δ

If μ is petite for δ and ψ_μ is the representation of W on $\mathrm{Hom}_M(\mu, \delta)$, then

$$\boxed{\mathcal{A}_\mu^G(w_0, \delta, \nu) \equiv A_{\psi_\mu}^{\mathbb{H}}(w_0, \tilde{\nu})}$$

The pseudospherical case

Similar to the spherical case:

- The stabilizer of δ is W.
- Every "α-factor" is an endomorphism of $\mathrm{Hom}_M(\mu,\delta)$.
- $\mathrm{Hom}_M(\mu,\delta)$ carries a repr. ψ_μ of W and an action of $\mathrm{d}\mu(Z_\alpha)^2$.
- Decompose $\mathrm{Hom}_M(\mu,\delta)$ in Z_α^2-eigenspaces:

$$\mathrm{Hom}_M(\mu,\delta) = \bigoplus_{l\in\mathbb{N}/2} E^\alpha(-l^2).$$

(Here $l \in 2\mathbb{N}+\frac{1}{2}$ if α is metaplectic, and $l \in 2\mathbb{N}$ otherwise.)

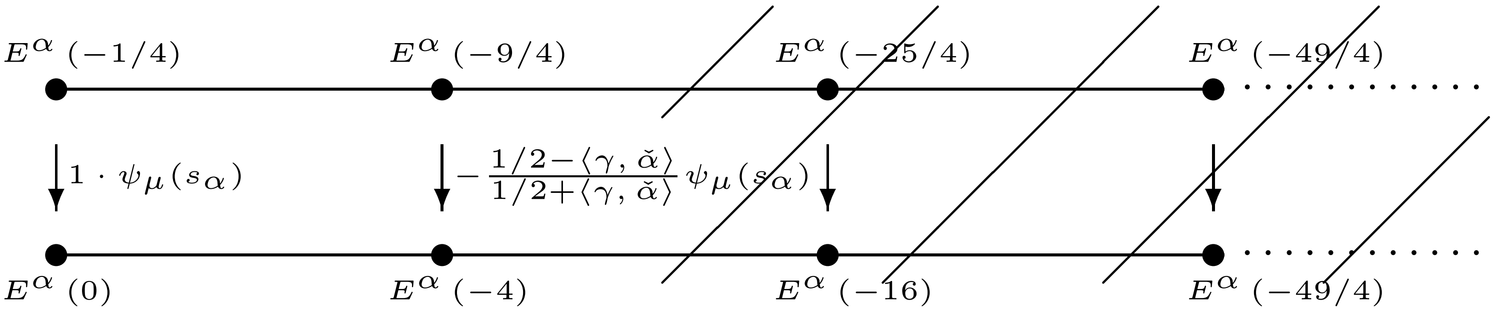
- $\mathcal{A}_\mu(s_\alpha,\rho,\lambda)$ acts on acts on $E^\alpha(-l^2)$ by

$$T \mapsto \underbrace{c(\alpha,\gamma,l)}_{a\ scalar}\underbrace{\mu_\delta(\sigma_\alpha)T(\mu(\sigma_\alpha)^{-1})}_{\psi_\mu(s_\alpha)T}.$$

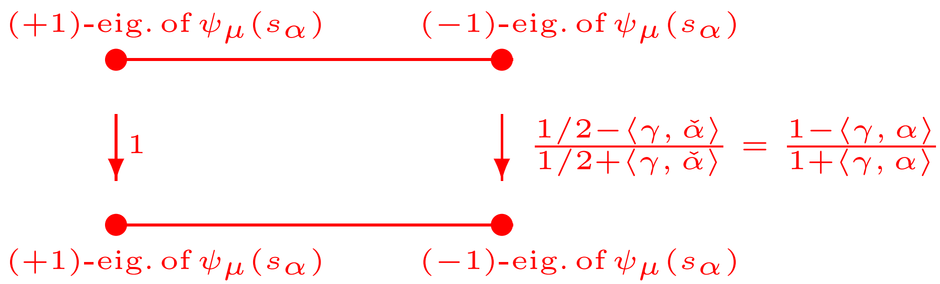
The scalar depends on whether α is metaplectic or not.

Petite K-types for δ pseudospherical (α metaplectic)

In this case, $(V_\mu^*)^M = E^\alpha(-1/4) \oplus E^\alpha(-9/4)$.



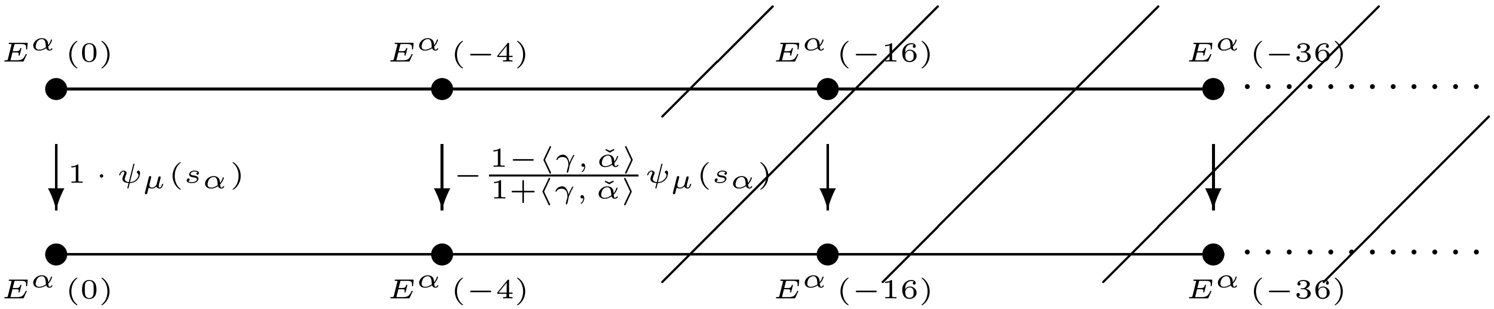
The spaces $E^\alpha(-1/4)$ and $E^\alpha(-9/4)$ coincide with the $(+1)$- and (-1)-eigenspace of $\psi_\mu(s_\alpha)$, respectively.



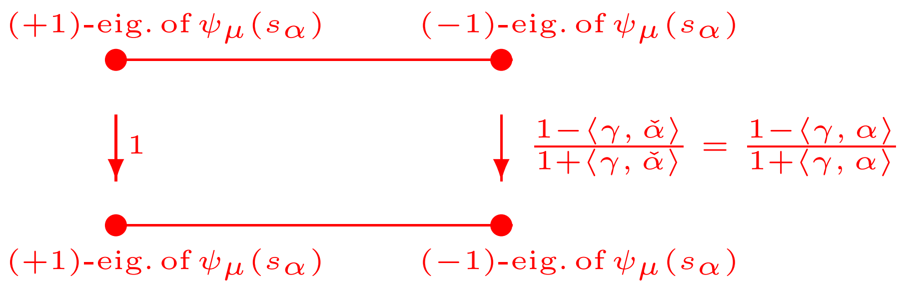
Hence $\mathcal{A}_\mu(s_\alpha, \delta_0, \lambda) = \frac{I+\langle\gamma,\alpha\rangle\psi_\mu(s_\alpha)}{1+\langle\gamma,\alpha\rangle}$. $\Longleftarrow$ *operator for* $\mathbb{H}(\Delta)$

Petite K-types for δ pseudospherical (α not metaplectic)

In this case, $(V_\mu^*)^M = E^\alpha(0) \oplus E^\alpha(-4)$.



The spaces $E^\alpha(0)$ and $E^\alpha(-4)$ coincide with the (+1)- and (−1)-eigenspace of $\psi_\mu(s_\alpha)$, respectively.



Hence $\mathcal{A}_\mu(s_\alpha, \delta_0, \lambda) = \frac{I+\langle\gamma,\alpha\rangle\psi_\mu(s_\alpha)}{1+\langle\gamma,\alpha\rangle}$. $\Longleftarrow$ *operator for* $\mathbb{H}(\Delta)$

The non-spherical non-pseudospherical case

In this case, *not* every reflection stabilizes δ.

◇ $W^\delta \subset W$ and $\Delta_\delta \subset \Delta$

◇ $\boxed{\mathbb{H}}$ = Hecke algebra with root system $\begin{cases} \Delta & \text{if } \delta \text{ genuine} \\ \check{\Delta} & \text{otherwise} \end{cases}$

Theorem. If μ is petite for δ and ψ_μ is the representation of W^δ on $\mathrm{Hom}_M(\mu, \delta)$, then

$$\boxed{\mathcal{A}_\mu^G(w_0, \delta, \nu) \equiv A_{\psi_\mu}^{\mathbb{H}}(w_0, \tilde{\nu})}$$

These operators decompose as a product of "α-factors", mimicking a minimal decomposition of w_0 in W and in W^δ respectively. Hence $\mathcal{A}_\mu^G(w_0, \delta, \nu)$ has more factors.

Choose a minimal decompositions of w_0 in W^δ:

$$w_0 = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_r}.$$

Every root s_β which is simple in both W^δ and W gives rise to

$$\boxed{\begin{array}{c}\text{a single}\\ \text{factor}\\ \text{of } \mathcal{A}_\mu^G(w_0,\delta,\nu)\end{array}} = \begin{cases} \frac{I+\langle\gamma,\beta\rangle\psi_\mu(s_\beta)}{1+\langle\gamma,\beta\rangle} & \text{if } \delta \text{ is genuine} \\ \\ \frac{I+\langle\gamma,\check\beta\rangle\psi_\mu(s_\beta)}{1+\langle\gamma,\check\beta\rangle} & \text{otherwise.} \end{cases}$$

Every root s_β which is simple in W^δ but not simple in W gives rise to

$$\boxed{\begin{array}{c}\text{a product}\\ \text{of factors}\\ \text{of } \mathcal{A}_\mu^G(w_0,\delta,\nu)\end{array}} = \begin{cases} \frac{I+\langle\gamma,\beta\rangle\psi_\mu(s_\beta)}{1+\langle\gamma,\beta\rangle} & \text{if } \delta \text{ is genuine} \\ \\ \frac{I+\langle\gamma,\check\beta\rangle\psi_\mu(s_\beta)}{1+\langle\gamma,\check\beta\rangle} & \text{otherwise.} \end{cases}$$

This product reflects a minimal factorization of s_β in W.

PART 4

What petite K-type do for us:

Embedding of unitary duals

A matching of petite K-types with relevant W^δ-types

If μ is petite, the correspondence

$$\mu \in \widehat{K} \to \psi_\mu = \mathrm{Hom}_M(\mu, \delta) \in \widehat{W^\delta}$$

gives rise to a matching of operators:

$$\mathcal{A}^G_\mu(w_0, \delta, \nu) \equiv A^{\mathbb{H}}_{\psi_\mu}(w_0, \tilde{\nu}).$$

Theorem. If set $\{\psi_\mu \colon \mu \in \widehat{K} \textit{ petite}\}$ includes all the *relevant* W^δ-types, then we obtain an embedding of unitary duals:

$$\boxed{\delta\text{-complementary series of } G} \subseteq \boxed{\text{spherical unitary dual of } \mathbb{H}}$$

$$||\qquad\qquad\qquad\qquad ||$$

$$\{\nu \colon L^G(\delta, \nu)\ \textit{unitary}\} \qquad \{\nu \colon L^{\mathbb{H}}(\nu)\ \textit{unitary}\}$$

An embedding of unitary duals

Suppose that for each **relevant** W^δ-type ψ, there is a **petite** K-type μ such that $\psi_\mu = \psi$. $(\star)$.

$$\boxed{L^G(\delta,\nu) \text{ is unitary}} \quad =============> \quad \boxed{L^{\mathbb{H}}(\tilde{\nu}) \text{ is unitary}}$$

$$\Updownarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Updownarrow$$

$\mathcal{A}^G_\mu(w_0,\delta,\nu)$ pos. semidef. $\forall \mu \in \widehat{K}$	$\Rightarrow$	$\mathcal{A}^G_\mu(w_0,\delta,\nu)$ pos. semidef. $\forall \mu$ petite	$\overset{(\star)}{\Rightarrow}$	$A^{\mathbb{H}}_\psi(w_0,\tilde{\nu})$ pos. semidef. $\forall \psi$ relevant	$\overset{(\diamond)}{\Leftrightarrow}$	$A^{\mathbb{H}}_\psi(w_0,\tilde{\nu})$ pos. semidef. $\forall \psi \in \widehat{W}$

$(\diamond)$ Relevant W-types detect unitarity for $\mathbb{H}$. *[Barbasch-Ciubotaru]*

Spherical unitary duals for split groups

Theorem [Barbasch] If $\delta = \delta_0$, the matching holds for all real split groups G. Let $\mathbb{H}$ be the affine graded Hecke algebra with *root system dual to the one of* G. Then

$$\boxed{\boxed{\text{spherical unitary dual of } G}} \quad \subseteq \quad \boxed{\boxed{\text{spherical unitary dual of } \mathbb{H}}}$$

$\uparrow$

this is an equality for classical groups

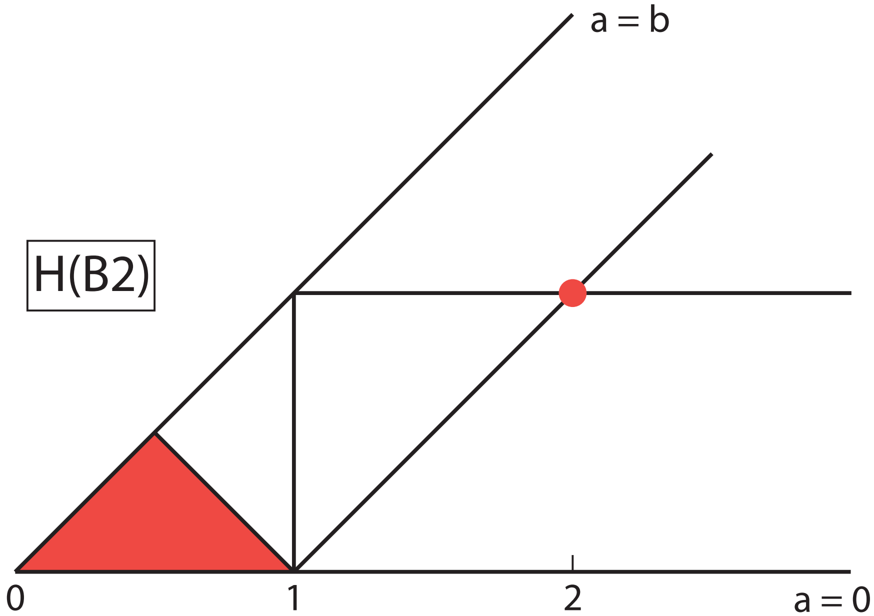
This gives us:

- the full spherical unitary dual of real split classical groups
- non-unitarity certificates for spherical representations of real split exceptional groups.

An example: the spherical unitary dual of $Sp(4)$

Let $\mathbb{H}$ be an affine graded algebra of type $B2$.

The spherical unitary dual of $Sp(4)$ embeds into the one of $\mathbb{H}(B2)$:



Actually, they are equal.

Pseudospherical unitary duals for split groups

Theorem [Adams, Barbasch, Paul, Trapa, Vogan] Let δ be pseudospherical. The matching holds for the double cover of every real split group of classical type. Let $\mathbb{H}$ be the affine graded Hecke algebra with *the same root system as* G. Then

$$\boxed{\text{pseudosph. unitary dual of } G} \quad \subseteq \quad \boxed{\text{spherical unitary dual of } \mathbb{H}}$$

$\uparrow$

this is an equality for $Mp(2n)$

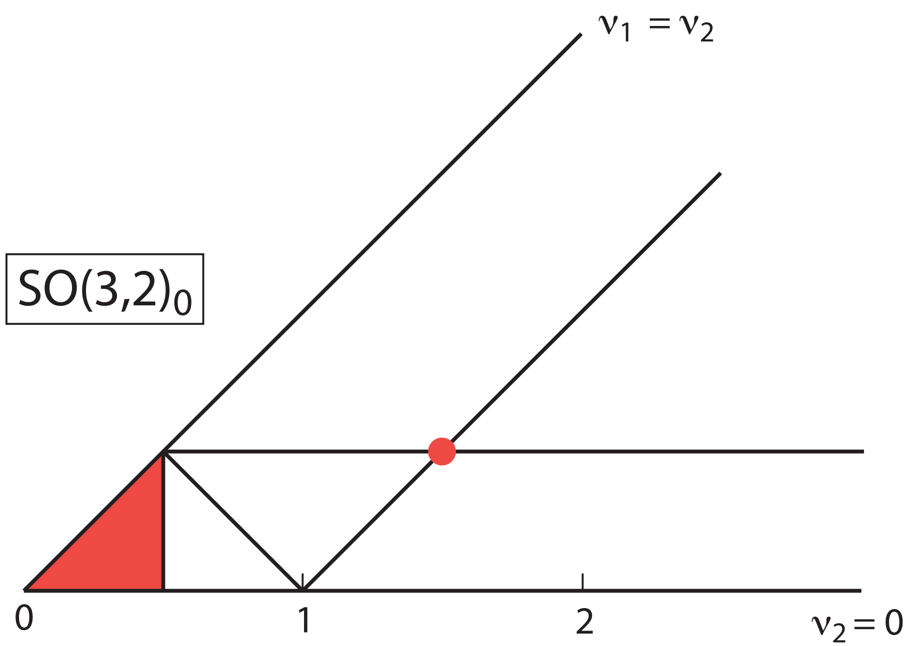
This gives us:

- the full pseudospherical unitary dual of $Mp(2n)$
- non-unitarity certificates for the other classical groups.

An example: the pseudospherical unitary dual of $Mp(4)$

Let $\mathbb{H}$ be an affine graded algebra Hecke of type $C2$.

The pseudospherical unitary dual of $Mp(4)$ coincides with the spherical unitary dual of $\mathbb{H}$ (which, in turn, coincides with the spherical unitary dual of of $SO(3,2)$).



Genuine complementary series of $Mp(2n)$

Genuine M-types of $Mp(2n)$ are parameterized by pairs of non-negative integers (p, q) with $p + q = n$. If $\delta = \delta^{p,q}$, then $\Delta_\delta = C_p \times C_q$.

Theorem [Paul, P., Salamanca] The matching holds for every genuine M-type $\delta^{p,q}$ of $Mp(2n)$. Let $\mathbb{H}$ be the affine graded Hecke algebra for the root system $C_p \times C_q$. Then

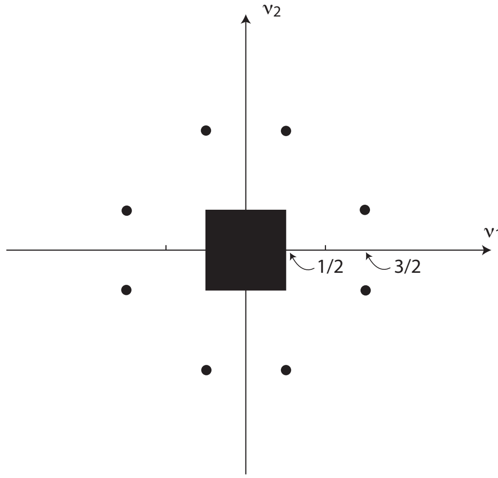
$$\boxed{\delta^{p,q}\text{-compl. series of } Mp(2n)} \subseteq \boxed{\text{spherical unitary dual of } \mathbb{H}}$$

$$\uparrow$$

an equality for $n \leq 4$ (all p, q), and for some special families of parameters (all n)

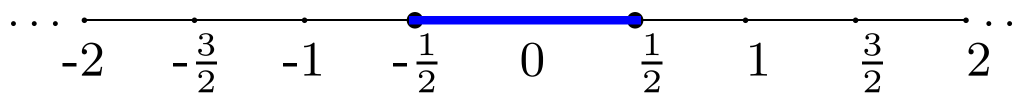
[Barbasch]: *spherical unitary dual of* $\mathbb{H}$ = *spherical unitary dual of* $SO(p+1,p) \times SO(q+1,q)$.

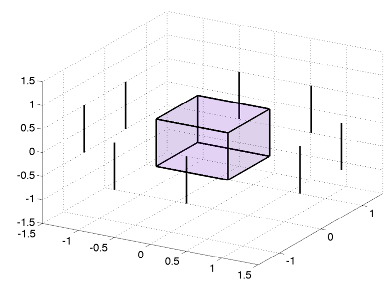
$$CS(Mp(6), \delta^{2,1}) = CS(SO(3,2)_0, \delta_0) \times CS(SO(2,1)_0, \delta_0)$$



$CS(SO(3,2)_0, \delta_0)$

$CS(SO(2,1)_0, \delta_0)$





$\boldsymbol{CS(Mp(6), \delta^{2,1})}$ **equals the product**

Complementary series of $SO(n+1,n)_0$

M-types of $SO(n+1,n)_0$ are parameterized by pairs of non-negative integers (p,q) with $p+q=n$. If $\delta=\delta^{p,q}$, then $\Delta_\delta = B_p \times B_q$.

Theorem [Paul, P., Salamanca] The matching holds for every M-type $\delta^{p,q}$ of $SO(n+1,n)_0$. Let $\mathbb{H}$ be affine graded Hecke algebra for the root system $C_p \times C_q$. Then

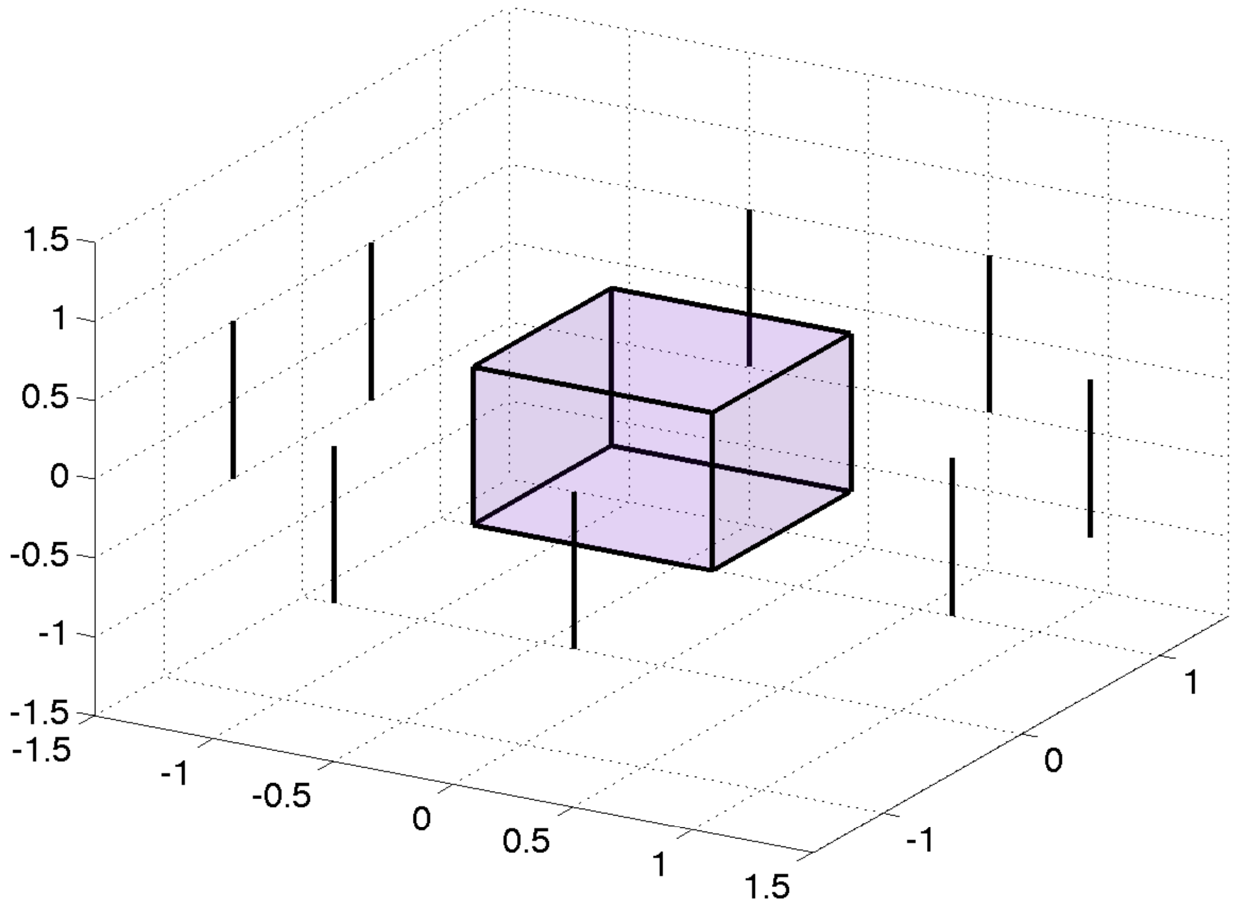
$$\boxed{CS(SO(n+1,n)_0, \delta^{p,q})} \subseteq \boxed{\text{spherical unitary dual of } \mathbb{H}}$$

$$\uparrow$$

an equality for $n \leq 4$ (all p, q), and for some special families of parameters (all n)

[Barbasch]: *spherical unitary dual of* $\mathbb{H}$ = *spherical unitary dual of* $SO(p+1,p) \times SO(q+1,q)$.

$$CS(SO(4,3)_0, \delta^{2,1}) = CS(SO(3,2)_0, \delta_0) \times CS(SO(2,1)_0, \delta_0)$$



The double cover of split $E6$

joint with D.Barbasch

repr. of $\tilde{M}$	**note**	**dim**	Δ_δ	*matching*
δ_1	trivial	1	$E6$	✓
$\boxed{\delta_8}$	pseudosph.	8	$E6$	*NO* $(\star)$
δ_{27}	non-genuine	$27 \cdot 1$	D_5	✓
δ_{36}	non-genuine	$36 \cdot 1$	A_5A_1	✓

$(\star)$ One relevant $W(E6)$-type is missing.

The double cover of split $F4$

joint with D.Barbasch

repr. of $\tilde{M}$	note	dim	Δ_δ	*matching*
δ_1	trivial	1	$F4$	✓
δ_2	pseudosph.	2	$F4$	✓
$\boxed{\delta_3}$	non-genuine	$3 \cdot 1$	$C4$	*No* $(\star)$
$\boxed{\delta_6}$	genuine	$3 \cdot 2$	$B4$	*No* $(\diamond)$
δ_{12}	non-genuine	$12 \cdot 1$	$B3 \times A_1$	✓

$(\star)$ One relevant $W(C4)$-type is missing.

$(\diamond)$ Two relevant $W(B4)$-types are missing.