March 15: David Vogan (MIT), "Characters, cohomology, and Kazhdan-Lusztig." FOLLOWED BY PIZZA IN 2-135.

This is the not-really-annual Free Pizza for Graduate Students Lie groups seminar. There won't be any mathematics interesting to the usual faculty suspects.

One of the most basic results about a semisimple Lie algebra g is the Weyl character formula. One way to think about it (as explained in Humphreys' book) is this. Start with a "triangular decomposition" $g = n^- + h + n$. Here h is a Cartan subalgebra of g and $b = h + n$ is a Borel subalgebra of g.

Now one can attach to each linear functional $\gamma \in h^*$ two modules for the Lie algebra g. The most interesting is the "irreducible highest weight module" $L(\gamma)$. The easiest is the "Verma module" $M(\gamma)$. Each of these modules decomposes under h into a direct sum of finite-dimensional weight spaces:

$$L(\gamma) = \sum_{\mu \in h^*} L(\gamma)(\mu),$$

and similarly for $M(\gamma)$. The fundamental problem of character theory is to compute $\dim(L(\gamma)(\mu))$ for every γ and μ. Here is a way to approach that problem.

The weight space dimensions for Verma modules are fairly accessible: there is an easy-to-compute integer-valued function P on h^* (the Kostant partition function) so that $\dim M(\gamma)(\mu) = P(\mu - \gamma)$. As is explained in Humphreys, every $L(\gamma)$ can be written as a finite integer combination of various Verma modules $M(\gamma')$:

$$L(\gamma) = \sum_{\gamma' \in h^*} c(\gamma', \gamma) M(\gamma').$$

Now we can write

$$\dim L(\gamma)(\mu) = \sum_{\gamma' \in h^*} c(\gamma', \gamma) P(\mu - \gamma').$$

So we know these dimensions as soon as we know the integers $c(\gamma', \gamma)$.

I'll explain what these integers have to do with Lie algebra cohomology; what this formalism has to do with the Weyl character formula; and finally how Kazhdan-Lusztig theory lets you compute the integers $c(\gamma', \gamma)$.