HYPERBOLIC PLANES AND WITT'S CANCELLATION THEOREM

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In this lecture, we will be introduce hyperbolic planes and prove a couple of results about them and their applications. Throughout these notes, it will be assumed that we are working over a field F with char $\neq 2$.

Recall from the lecture on bilinear forms, that a *bilinear form* B on V is a function $B : V \times V \mapsto F$ which becomes linear in one variable if the other variable is fixed. Any bilinear form has a corresponding representational matrix $\hat{B}$ relative to a basis $\{v_i\}$ of V. The matrix consists of entries $b_{ij} = B(v_i, v_j)$ for all i,j. Also, B is called

- *symmetric* if $B(v, w) = B(w, v)$ for all $v, w \in V$;
- *alternate* if $B(v, v) = 0$ for all $v \in V$;
- *reflexive* if $B(v, w) = 0 \implies B(w, v) = 0$. (Proposition 2.7 in the text also shows that B is reflexive iff it is either symmetric or alternate).

Hyperbolic Planes: Case of an Alternate Bilinear Form on V

Definition. Given a vector space V with an alternate bilinear form B on V, we call a pair of vectors $u, v \in V$ a *hyperbolic pair* if $B(u, v) = 1$. Consequently, the subspace $H \in V$ spanned by (u, v) will be called a *hyperbolic plane.*

We can construct a hyperbolic plane in V by choosing a hyperbolic pair (u, v) in the following way: for some linearly independent $\hat{u}, \hat{v} \in V$ such that $B(\hat{u}, \hat{v}) = b \neq 0$, set $u = b^{-1}\hat{u}$, and $v = \hat{v}$. Notice that $B(u,u) = 0$ and $B(v,v) = 0$ as well as $B(u,v) = B(b^{-1}\hat{u},\hat{v}) = b^{-1}B(\hat{u}.\hat{v}) = b^{-1}b = 1$. Thus, (u,v) is a hyperbolic pair in V and the space $H = \langle u,v\rangle$ is a hyperbolic plane. Moreover, as the following theorem shows, V can be represented using a set of mutually orthogonal hyperbolic planes.

Theorem. *If B is an alternating form on V, then*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r \oplus \mathrm{rad}V,$$

a direct sum of mutually orthogonal subspaces with each W_i a hyperbolic plane.

Proof. The proof will proceed by induction on dimension n of the space V.

If B=0, then rad$V = V$ and we are done.

Suppose the result is true for spaces of dimension less than or equal to n. Consider a space V with dimension $n+1$ and the same bilinear form B. Choose a hyperbolic pair (u,v) in V as described above. Let the hyperbolic plane spanned by u,v be W_1. The restriction matrix of B to subspace W_1, relative to the basis (u,v), is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since the determinant of this matrix is 1, then discr$B\ |_{W_1} \neq 0$, and thus W_1 is a nondegenerate subspace of V relative to B. Proposition 2.9 of the text states that for a reflexive form B on space V and nondegenerate subspace W, V is a direct sum of W and its complement. This proposition applies to B since alternate bilinear forms are also reflexive, and we get that $V = W_1 \oplus {W_1}^{\perp}$. Since $\dim(W_1^{\perp}) = \dim(V) - \dim(W_1) = n-1$, the subspace $W_1^{\perp}$ can be represented as a direct sum of mutually orthogonal hyperbolic planes and rad$(W_1^{\perp})$.

Notice that, conveniently, rad$V = V^{\perp} = (W_1 \oplus W^{\perp})^{\perp} = W_1^{\perp} \cap {W_1^{\perp}}^{\perp} = \mathrm{rad}(W_1^{\perp})$. Therefore, we now have a representation of $n+1$-dimensional space V as a direct sum of mutually orthogonal hyperbolic planes and rad$V$. By induction, the statement of the theorem is true.

Hyperbolic Planes: Case of a Symmetric Bilinear Form on V

Of course we can extend the definition of a hyperbolic plane to vector spaces with bilinear forms that are not alternate. In particular, let us consider a vector space V with a bilinear form B that is symmetric. Since having an alternate bilinear form guaranteed that for $u, v \in V$, we have $B(u, u) = B(v, v) = 0$, now we will have to impose this condition on basis vectors of a hyperbolic plane in a different way.

Definition. Given a vector space V with a symmetric bilinear form B, define *quadratic form* Q corresponding to B as a function $Q : V \mapsto F$ with $Q(v) = B(v, v)$.

It is easy to see that for all $a \in F$ and $v \in V$, we get $Q(av) = a^2Q(v)$ - hence the name of the form. Also, one should keep in mind that even though we define Q in as a quadratic form associated with B, the bilinear form B is completely determined by Q: $B(u, v) = \frac{1}{2}[Q(u + v) - Q(u) - Q(v)]$. (This expression is derived by evaluating $Q(u + v)$ using the above definition and then solving the resulting equation for $B(u, v)$.) Now, we are ready to give the definition of a hyperbolic plane in a space with a symmetric form.

Definition. Given B a nondegenerate symmetric bilinear form on V, for $u, v \in V$, we call (u, v) a *hyperbolic pair* if $B(u, v) = 1$, and $Q(u) = 0$, $Q(v) = 0$. The subspace of V spanned by a hyperbolic pair is a *hyperbolic plane*.

A space with such bilinear form, i.e. with a nondegenerate symmetric bilinear form, is called a *quadratic space*. This is the terminology we will use throughout the rest of the notes. (Note: we need the bilinear form to be nondegenerate in order to make sure that $B(u,v) = 1$. Allowing the bilinear form to have zero determinant of the corresponding matrix $\hat{B}$, would also allow $\hat{B}$ to have all diagonal entries as 0. In such case, $B(v_i, v_i) = Q(v_i) = 0$ for all i, and hence $B(u,v) = 0$ for all $u, v \in V$. We can also see that nondegeneracy of the form restricted to the hyperbolic plane $\langle u, v\rangle = H$ is implied from $\hat{B}_H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, whose determinant is 1.)

Witt's Cancellation Theorem

Definition. If V is a vector space with quadratic form Q, then vector $v \in V$, $v \neq 0$ is called

- *isotropic* if $Q(v) = 0$;
- *anisotropic* if $Q(v) \neq 0$. (Note: $v = 0$ is also taken to be anisotropic.)

Space V is called *isotropic* if it contains at least one isotropic vector. If all vectors in V are isotropic, we will say that V is *totally isotropic*.

Definition. Given V, a quadratic space with bilinear form B, an *isometry* of V is defined to be any linear transformation $\sigma : V \mapsto V$ such that for all $u, v \in V$, $B(\sigma u, \sigma v) = B(u, v)$.

Theorem. *(Witt's Cancellation Theorem). Suppose that U_1 and U_2 are nondegenerate subspaces of a quadratic space V and that $\sigma : U_1 \mapsto U_2$ is an isometry. Then $U_1^\perp$ and $U_2^\perp$ are also isometric.*

Proof. The proof will proceed by induction on the dimension of U_1.

For base case, suppose that $U_1 = \langle u_1\rangle, U_2 = \langle u_2\rangle$, that is $\dim U_1 = 1$. Since U_1 and U_2 are nondegenerate subspaces, $Q(u_1)$ and $Q(u_2)$ are nonzero. Without loss of generality, assume that $\sigma u_1 = \sigma u_2$. Then, $Q(u_1) = Q(u_2)$. It is not difficult to verify that the equality $Q(u_1 \pm u_2) = 2Q(u_1) \pm 2B(u_1, u_2)$ implies that only one of

$Q(u_1+u_2)$ and $Q(u_1-u_2)$ can be 0. Thus, at least one of them, say $Q(u_1-u_2)$ is nonzero. (The case that is worked out in the book is $Q(u_1+u_2)\neq 0$, and it is very similar.)

Since $Q(u_1)=Q(u_2)$, and $B(u_1+u_2, u_1-u_2)=Q(u_1)-Q(u_2)=0$, then $(u_1-u_2)\perp(u_1+u_2)$. Hence, $\sigma_{u_1-u_2}(u_1+u_2)=(u_1+u_2)$ (*). Notice that we can represent u_1 the following way: $u_1=\frac{1}{2}(u_1+u_2)+\frac{1}{2}(u_1-u_2)$. Using equation (*) and this representation of u_1, we get that $\sigma_{u_1-u_2}(u_1)=\sigma_{u_1-u_2}([frac12(u_1+u_2)]+[frac12(u_1-u_2)])=[\frac{1}{2}(u_1+u_2)]-[\frac{1}{2}(u_1-u_2)]=u_2$. And in general, $\sigma_{u_1-u_2}(\langle u_1\rangle)=\langle u_2\rangle$. Since σ is an isometry, then by definition, $B(\sigma(e\langle u_1\rangle),\sigma(\langle u_1\rangle^\perp))=B(\langle u_1\rangle,\langle u_1\rangle^\perp)=0$, i.e. $\sigma(\langle u_1\rangle^\perp)=(\sigma(\langle u_1\rangle))^\perp=\langle u_2\rangle^\perp$, and thus $U_1^\perp$ and $U_2^\perp$ are isometric for our base case of $\dim U_1=1$.

Assume that the theorem is true for U_1 of dimension less than or equal to n. Consider a nondegenerate subspace U_1 of V, $\dim U_1=n+1$. Pick $u\in U_1$, an anisotropic vector. Let the subspace of $\langle u^\perp\rangle$ in U_1 be W_1. Since $\langle u\rangle$ is nondegenerate, then $U_1=\langle u\rangle\oplus W_1$. Let $\sigma u_1=u_2$, and $\sigma W_1=W_2$, then since σ is an isometry, u_2 is anisotropic and hence $\langle u_2\rangle$ is nondegenerate and $U_2=\langle u_2\rangle\oplus W_2$. Thus we can represent V in two ways: $V=(\langle u_1\rangle\oplus W_1)\oplus U_1^\perp=\langle u_2\rangle\oplus(W_2\oplus U_2^\perp)$ and $V=(\langle u_2\rangle\oplus W_2)\oplus U_2^\perp=\langle u_2\rangle\oplus(W_2\oplus U_2^\perp)$. Using the one dimensional case, we know that there exists some isometry, call it η, such that $\eta: W_1\oplus U_1^\perp\mapsto W_2\oplus U_2^\perp$, that is $\eta W_1\oplus\eta U_1^\perp=W_2\oplus U_2^\perp$. Since $W_2=\sigma W_1$, then $\eta\sigma^{-1}: W_2\mapsto\eta W_1$ is an isometry. Notice that $\dim W_2\langle\dim U_1$ and so the induction assumption implies that since W_2 and ηW_1 are nondegenerate subspaces of $W_2\oplus U_2^\perp$, and since there exists an isometry, namely $\eta\sigma^{-1}$, that maps from one subspace to the other, then their orthogonal complements, $U_2^\perp$ and $\eta U_1^\perp$ respectively, must be isometric. It is easy to see that an isometry mapping $U_2^\perp$ to $\eta U_1^\perp$ can be multiplied by η^{-1} to get an isometry from $U_1^\perp$ to $U_2^\perp$. This completes the induction step and the proof of Witt's Cancellation Theorem.

Definition. The dimension m of a maximal totally isotropic subspace of a quadratic space V is called the *Witt index of* V. (Note: Witt index is well defined since the Corollary of Witt's Extension Theorem presented on page 41 of the text states that any two maximal totally isotropic subspaces of V have the same dimension, and moreover every totally isotropic subspace is contained in one of maximal dimension.)

Definition. A subspace H of a quadratic space V is called a *hyperbolic subspace* if H is a direct sum of mutually orthogonal hyperbolic planes.

Theorem. *(Theorem 5.4 of the text.) If V is a quadratic space with Witt index m, then V has a hyperbolic subspace H of dimension $2m$ and an anisotropic subspace X with $V = H \oplus X$. If $V = H' \oplus Y$ is any orthogonal splitting with H' hyperbolic and Y anisotropic, then $\dim H' = 2m$, and Y and X are isometric.*

We will not present the proof of this theorem, however it is easy to see that demonstrating Y and X are isometric will involve use of Witt's Cancellation Theorem. Notice also that based on this theorem we can define Witt index m as the maximum number of mutually orthogonal hyperbolic planes in V, since if we construct a basis using vectors u_i taken from hyperbolic pairs (u_i, v_i) spanning each of the m hyperbolic planes, then we will get a basis of size m for a maximal totally isotropic subspace.

Corollary. *If V has dimension n and Witt index m then $m \leq n/2$.*

Figure 1. A crocheted model of a hyperbolic plane

Appendix

Even though we have developed some feel for the geometry of hyperbolic planes, visualizing one might not be easy. The link included at the end of this section is a link to a fun article about constructing physical models of a hyperbolic plane. The article is actually an interview with Daina Taimina, a mathematician at Cornell University, who crochets models of a hyperbolic plane, one of which is shown in Figure 1.

⟨ http://www.cabinetmagazine.org/issues/16/crocheting.php ⟩