CORRECTIONS FOR *LECTURES ON THE ORBIT METHOD*, BY A.A. KIRILLOV.

Version of January 1, 2005

This is a list of corrections for the book *Lectures on the Orbit Method*, by A.A. Kirillov (Graduate Studies in Mathematics, Volume 64, American Mathematical Society, 2004). Because the book is aimed at students, I have not omitted "trivial" or "obvious" corrections; my own experience with books in areas unfamiliar to me is that there is no such thing as a trivial mistake. I am an ardent admirer of the orbit method and of its creators, so I want to support students reading this book. They are they intended audience of these notes.

This list is certainly incomplete, and it certainly includes corrections that are inadequate. I have little doubt that it includes corrections of statements that were not wrong. I would be very pleased to have errors of all three sorts brought to my attention: I will try to update the list accordingly.

References from Kirillov's bibliography are cited as **[Di3]**. Additional references at the end of this list are [S].

page 3, Example 2. It is not true that an $SO(n,\mathbb{R})$ orbit on skew-symmetric matrices is determined by its spectrum. The possible eigenvalues are as stated in the text: multisets of n elements in $i\mathbb{R}$, symmetric with respect to complex conjugation. If n is even, and all the eigenvalues are non-zero, then there are *two* orbits having a specified set of eigenvalues. For example, there are two ($SO(2,\mathbb{R})$-orbits of) skew-symmetric real matrices with eigenvalues i and $-i$:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In all other cases (equivalently, if at least one of the eigenvalues is zero) the orbit is determined by the eigenvalues.

If the connected group $SO(n,\mathbb{R})$ is replaced by the disconnected group $O(n,\mathbb{R})$ (of all real orthogonal matrices), then the orbits of $O(n,\mathbb{R})$ are in one-to-one correspondence with the multisets of eigenvalues.

The collection of real numbers $\{\operatorname{tr} X^2, \dots, \operatorname{tr} X^{2k}\}$ (with $k = \left[\frac{n}{2}\right]$) precisely determines the multiset of eigenvalues; so it does not determine the orbit when n is even.

page 8, proof of Theorem 2. The argument given shows that the coadjoint orbit Ω_F is open in the symplectic leaf L_F. To complete the proof, notice that L_F must be the disjoint union of the coadjoint orbits that it contains. Because each orbit is open, each (as the complement of the union of the remaining orbits in L_F) must also be closed. Now L_F is (by definition) connected, and Ω_F is a non-empty open and closed subset; so $L_F = \Omega_F$.

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page 9, line 3. The reference to Appendix II.3.2 should be II.3.4.

page 14, Proposition 3. The result stated here is difficult to interpret precisely, and many possible interpretations are false. The references cited are [**Bor**] and [**R**]. As far as I can tell, there is nothing close to this result in Borel's text [**Bor**]. Richardson in [**R**] proves something like this in an extremely special case, but he does not proceed by way of a general result close to this. It is possible that the reference [**R**] was intended to be [Ro]. Theorem 2 of [Ro] (on page 407) implies something close to Proposition 3: the main difference is that one has to replace X by a G-invariant open subset $U \subset X$.

Here is one of the difficulties. A rational invariant is by definition a quotient of two polynomial invariants; so it has a well-defined value only where the denominator is non-zero. To speak of "level sets of rational invariants" therefore makes sense only where all denominators are non-zero. In most examples, the union of the zeros of denominators of *all* rational invariants is all of X, so this is clearly not what is meant. What is reasonable is to choose some finite generating set of rational invariants, and to exclude the zeros of their finitely many denominators. In order for this approach to make sense, one needs the field of rational invariants to be finitely generated. This is true, and not very difficult. I do not know a good reference, but it follows from Theorem 2 of [Ro].

To see some of the simple difficulties with Proposition 3, let $X = \mathbb{C}^2$ and

$$G = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a \in \mathbb{C}^\times, b \in \mathbb{C} \right\}.$$

The open set of points $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x \neq 0$ is a single orbit of G; every point on the y-axis is fixed by G, and is therefore an orbit. The only rational invariants are therefore the constant functions. The "common level set of all rational invariants" is all of X, which consists of infinitely many orbits. To get something like Proposition 3 to be true, one has to replace X by the complement U of the y axis. This is a smaller open set than is forced on us by vanishing of denominators.

Some more subtle difficulties of the same nature can be seen in the example studied in pages 79–80 of the text.

There is a fairly nice general relationship between orbits and invariants in the case of an algebraic action of a *reductive* group G on an affine algebraic variety. The basic reference is [M]; there is a nice introduction in Chapter 3 of [N].

page 21, Remark 4. The internal reference should be to Examples 6 and 7.

page 24, Lemma 8. This is false without some additional hypothesis, for example that the foliation be a fibration. The reason is that otherwise $C_P^\infty(M)$ is too small. The mistake in the proof is in the last line (lines 4–5 on page 25): the space of skew gradients of functions in $C_P^\infty(M)$ need not have dimension $\frac{1}{2}\dim M$. For example, suppose M is the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$, with the symplectic structure inherited from the standard one on $\mathbb{R}^2$. Let P be the polarization generated by the vector field

$$\xi = \partial/\partial x + \sqrt{2}\partial/\partial y;$$

what matters about the constant $\sqrt{2}$ is that it is irrational. Functions in $C_P^\infty(M)$ are constant on the integral curves of ξ, which are the lines

$$\{(a,b) + t(1,\sqrt{2}) \mid t \in \mathbb{R}\}$$

for fixed $(a,b) \in \mathbb{R}^2$. Each such line is dense in the torus M, so a function constant on the lines must be constant. That is,

$$C_P^\infty(M) = \text{constant functions on } M.$$

The space of skew-gradients of these functions has dimension 0 (not 1). We can adjoin to the constants any non-constant smooth function f on M, obtaining a two-dimensional abelian subalgebra that properly contains $C_P^\infty(M)$.

page 27, line 3 (formula (ii)). The formula for the dimension of a polarization $\mathfrak{h}$ (which is one of the most important in the orbit method) is incorrect. The line should read

$$(\textit{i.e.}, \mathfrak{h} \text{ has maximum possible dimension } \frac{\dim \mathfrak{g} + \dim stab(F)}{2})$$

The formula in the text is correct for generic elements $F \in \mathfrak{g}^*$.

page 29, Theorem $5'$. The hypothesis of the theorem incorrectly omits the word "admissible": it should read in part

set of all admissible complex algebraic polarizations $\mathfrak{h}$

page 44, proof of Theorem 3. The language used in the formulation of the italicized assertion may be misleading. What is meant is

If all of the eigenvalues of a diagonal matrix T are distinct, then any matrix A that commutes with T is itself a diagonal matrix.

page 48, line 8. The reference to formula (23) should be to (18).

page 48, lines 9–11. These assertions are incorrect in detail. The operators u and v are unitary operators on the Hilbert space $\mathcal{H}$ of the canonical commuation relations (as explained before (18) on page 42). They commute with each other, and therefore the collection of Laurent polynomials $\sum_{m,n\in\mathbb{Z}} c_{m,n} u^m v^n$ is a commutative algebra of operators on $\mathcal{H}$. The term "Laurent polynomials" can only mean that this sum is intended to be finite.

Since the operators u and v are unitary, the analogous formal infinite sum is convergent in the operator norm as long as the series $\sum |c_{m,n}|$ converges, and the resulting collection of operators is commutative. In order to get a *maximal* commutative algebra of operators, one must adjoin to the Laurent polynomials at least all of these infinite sums. (Probably one actually needs to use formal sums of operators with $c_{m,n}$ the Fourier coefficients of any L^∞ function on the torus; I have not checked this carefully.)

In order to get an algebra that is "isomorphic to the algebra of smooth functions on the two-dimensional torus," one should use rapidly decreasing coefficients:

$$|c_{m,n}| \leq A_k(m^2+n^2+1)^{-k}, \qquad \text{all } k \geq 0.$$

This is intermediate between the Laurent polynomials and the maximal commutative algebra.

page 72, Proposition 2(d). This is false unless G is simply connected. The simplest example is G equal to the one-dimensional circle group, which is abelian and therefore (according to Definition 2) nilpotent. The group of upper triangular matrices with ones on the diagonal has no non-trivial compact subgroups, so it cannot contain a copy of G.

page 74, line 5. The formula for the dimension of a polarization is incorrect. The correct formula is

$$\dim \mathfrak{h} = \frac{\dim G + \dim StabF}{2}.$$

page 74, after (6). I do not know a precise statement or reference for the "Frobenius Duality Principle."

pages 74–75 sentence "If we fix a smooth measure . . . " This is false for most choices of smooth measure. Choosing a Haar measure identifies generalized functions with distributions; in this identification, generalized characters are indeed tempered distributions.

page 75, lines 3–7. I do not know what is meant by "regular distribution," and I cannot find the term elsewhere in the text. In order for these assertions to make sense, "regular distribution" should mean "integration against a continuous function times a smooth density." The assertion that "any distribution is a derivative of a regular distribution" is then precisely true for distributions of compact support, and locally true for any distribution. It seems reasonable to guess that it is precisely true for the character of an irreducible representation of a nilpotent Lie group, but I do not know a proof.

page 75, Definition 2.6. This definition of "functional dimension" certainly applies only in the nilpotent case, and using it even there requires great care. Notice for example that (28) on page 48 identifies the Schwartz space of the one-dimensional space $\mathbb{R}$ with smooth sections of a line bundle on the two-dimensional torus $\mathbb{T}^2$.

page 76, after (7). The assertion that "GK-dimension generalizes the notions of Krull dimension for commutative rings and of transcendence degree for quotient fields" requires some qualification and explanation. Suppose that A is a finitely generated commutative algebra over a field k (that is, a quotient of a polynomial ring over k). Then the Krull dimension of A (defined to be the maximum length of a strictly increasing chain of prime ideals in A) is equal to its GK-dimension.

This is a consequence of the theory of Hilbert functions for local rings (Chapter 11 of [AM], for example). If A is a domain, then this is equal to the transcendence degree of the quotient field of A over k ([AM], Theorem 11.25).

page 77, footnote 2. The reference to **[Di1]** is not very helpful, since this bibliographical entry consists of six articles covering about 170 pages. The reference here should probably be to Dixmier's article III.

page 110, Proposition 1. The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ satisfies condition (b) but not the (supposedly equivalent) condition (a). (The exponential map for the corresponding simply connected group is one-to-one but not onto: for the group $SL(2,\mathbb{R})$, the matrix

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not in the image of the exponential map.) The proposition should either assume at the beginning that G is solvable, or replace (b) by

b) *The Lie algebra* $\mathfrak{g}$ *has no subalgebra isomorphic either to* $\mathfrak{e}(2)$ *or to* $\mathfrak{sl}(2,\mathbb{R})$.

pages 123–124, definition of rigged. The Auslander-Kostant Theorem 3, parametrizing irreducible unitary representations in terms of rigged coadjoint orbits, makes sense as stated also for type I solvable groups that are not simply connected. I believe that it is false in that setting (but I do not know an example or a reference). This fact, and parallel problems in the case of compact groups, suggest that the notion of rigged orbit is not yet the right one. Duflo's notion [D] of *admissible orbit datum* addresses these objections: it allows for generalizing Auslander-Kostant, and it works well for compact groups. (This is a criticism, not a correction.)

page 126, (20). This formula (realizing the diamond Lie algebra as a subalgebra of a symplectic Lie algebra) is correct, but it takes place in an unusual version of the symplectic Lie algebra: not the one described in most standard texts. A discussion of this text's version of the symplectic Lie algebra appears in the note for page 315 below.

page 127, (23c). The formula should read

$$\Omega_0^c = (c, 0, 0, 0), \quad c \in \mathbb{R}.$$

The corresponding one-dimensional representations have the three-dimensional normal subgroup with Lie algebra spanned by X, Y, and Z acting trivially, and T acting by $2\pi i c$.

page 133, Theorem 4. The theorem proved in [Sh] assumes also that H is connected. It is conceivable that the theorem is true without that assumption, but one would need a substantially more complicated proof.

page 133, definition of basic. The definition is incorrect; as stated it allows H to be trivial, in which case K is also trivial, so $G/K = G$ can be any solvable group. Here is the correct version, taken from [Sh].

Suppose G is a connected and simply connected type I solvable Lie group, and H is a *maximal* closed connected subgroup of type I. Let K be the largest connected normal subgroup of G contained in H. Then the quotient group G/K is called a **basic group**.

page 133, Lemma 3.2. The end of the sentence should read, "multiplications by $e^{\gamma t}$, $\gamma \in \mathbb{C}\backslash\mathbb{R}$." This means the one parameter subgroup consisting of all these multiplications, as t varies over $\mathbb{R}$.

The connection between Kirillov's notation and that in [Sh] is slightly confusing. Shchepochkina's Lie algebra $\mathcal{G}_3(\alpha)$ (defined in [Sh, page 330]) is the Lie algebra of Kirillov's group $G_3(\alpha+i)$. Replacing γ by a non-zero real multiple does not change $G_3(\gamma)$, so every non-real γ is accounted for.

page 133, solution to Exercise 10(c). This is a very interesting example, and the solution given is incorrect. The induced representation is a discrete direct sum of continuous direct integrals of the representations π_{c_1,c_2} with $c_1c_2 > 0$ and $2\pi c_2 \in \mathbb{Z}$. The integrality condition is omitted in the text.

The error arises from a subtlety in the definition of "lying over" for rigged moments (equation (30) on page 132). All of the orbits Ω_{c_1,c_2} with $c_1c_2 > 0$ lie over the zero orbit for the group $H = \exp(\mathbb{R}T)$. That is, we can always choose a point $F \in \Omega_{c_1,c_2}$ lying over $F' = 0$. This is condition (30)(a). Write χ and χ' for the unique rigged momenta at F and F'; these are representations of $Stab(F)$ and $Stab(F') = H$ respectively.

Condition (30)(b) is that

$$\chi = \chi' \text{ on } H \cap Stab(F).$$

Now $Stab(F)$ is a two-dimensional abelian subgroup of G, isomorphic to $\mathbb{R}^2$. If $c_2 \neq 0$, it is easy to check that T does not belong to the Lie algebra of $Stab(F)$. That is, $H \cap Stab(F)$ has trivial Lie algebra. For simply connected nilpotent groups, this circumstance would force $H \cap Stab(F)$ to be trivial, and condition (b) would automatically be satisfied. But in this case, it turns out that

$$H \cap Stab(F) = \{\exp(2\pi mT) | m \in \mathbb{Z}\} \simeq \mathbb{Z}.$$

(These elements belong to $Stab(F)$ even though T is not in the Lie algebra of $Stab(F)$.) Further calculation shows that

$$\chi(\exp(2\pi mT)) = \exp(2\pi i m(2\pi c_2)), \qquad \chi'(\exp(2\pi mT)) = 1.$$

Therefore condition (30)(b) amounts to $2\pi c_2 \in \mathbb{Z}$, the omitted integrality condition in the text.

page 135, paragraph 2. The assertion that "most abelian Lie groups are of the form $\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^l \times F$" is confusing. What is true is that an abelian Lie group A has the suggested form if and only if the group of connected components A/A_0 is finitely generated.

page 135, last paragraph. The restriction to connected and simply connected K eliminates almost all of the most familiar examples: the circle group $\mathbb{T}$, the unitary group $U(n)$, the orthogonal groups $O(n)$ and $SO(n)$. Most of the results in the text can be recast to cover these examples, but this is not a reasonable exercise for the reader. (This is a criticism, not a correction.)

page 135, bottom line. The internal reference to Appendix III.2.3 should be to Appendix III.3.4.

page 137, Chevalley basis. The characteristic properties of a Chevalley basis are stated incorrectly, more or less because of some mistakes on page 312 in Appendix III.3.3. Condition (ii) should be dropped. Condition (iii) should be

$$[X_\alpha, X_{-\alpha}] = H_\alpha.$$

Here $H_\alpha \in \mathfrak{h}$ is the coroot for α, characterized by

$$\lambda(H_\alpha) = 2(\lambda, \alpha)/(\alpha, \alpha).$$

To describe a Chevalley basis as "canonical" is misleading. If $\{X_\alpha\}$ is one such basis (corresponding to a fixed maximal torus $T \subset K$), there are two ways to construct others. One is to take an element $t \in T$ and define the new basis

$$X'_\alpha = \mathrm{Ad}(t)X_\alpha.$$

Another is to choose for every root α a sign $\epsilon_\alpha = \pm 1$, subject to the condition that $\epsilon_{-\alpha} = \epsilon_\alpha$. Then one gets a new Chevalley basis

$$X'_\alpha = \epsilon_\alpha X_\alpha.$$

These two operations (on Chevalley bases attached to T) commute with each other; by performing them in succession, one can get any such basis from the initial one. (This uniqueness result is a fairly easy exercise. What requires care is the existence.) The text gives no reference; one source is **[Hu]**, Theorem 25.2.

page 139, third paragraph. The lattices P and Q are defined on page 307.

page 142, second paragraph. To prove that every integral weight is the highest weight of a representation, the text says that "the explicit construction is usually used." Cartan's Theorem (Theorem 3 on page 157) is mentioned as an alternative approach. My understanding is that Cartan proved his theorem by the "explicit construction" of representations, so this does not appear at first to be an alternative. However, the text gives a proof of Cartan's theorem without recourse to case-by-case construction; the key step is in the next to last paragraph of page 159. So the "alternative" is genuine.

On the other hand, it is certainly no longer reasonable to say that "the explicit construction is usually used." Harish-Chandra gave the first proof of Cartan's theorem without case-by-case considerations in the early 1950s, using the "Verma modules" that he invented for this purpose. Harish-Chandra's proof is the one given in the standard text **[Hu]**.

page 148, line 1. I don't know a reference or a statement for the "well-known theorem about compact transformation groups" cited here. In any case the theorem is not used below.

page 148, Proposition 3(b). It is true that there are finitely many subgroups intermediate between T and K, but entirely false that each of them is the stabilizer of some point of $\mathfrak{t}$. There are two kinds of reasons. First, the stabilizer of an element of $\mathfrak{t}$ must be connected, and there are many disconnected subgroups containing T (like the normalizer of T).

One may deduce from reading the proposed proof that the author intended to say that any *connected* subgroup S between T and K must be the stabilizer of a point of $\mathfrak{t}$. This also is false, however. The simplest example is $K = SO(5)$, $T = SO(2) \times SO(2)$, $S = SO(4)$.

The gap in the proof appears in the fourth paragraph of page 149, where the existence of an appropriate element $\mu^{(m)}$ of $\mathfrak{t}$ is simply asserted without proof.

page 149, line 1. The symbol $\bigcap$ is extraneous and should be omitted.

page 156, four lines from bottom. The reference to a formula in the Appendix should be (50). In the following display, replace G by K (twice).

page 157, line 2. The reference to [**B**] is incorrect. Perhaps what is intended is [**Bou**], where there are related results in §5 and §6 of Chapter 5.

page 157, Theorem 3(a). I could not locate the definition of the partial order in Appendix III.3.1. It is

$$\lambda \geq \lambda_1 \iff \lambda - \lambda_1 \text{ is a sum of positive roots (repetitions allowed).}$$

page 159, proof of Proposition 4. In the fourth line, the formula should be $\Pi \simeq \pi \otimes \pi^*$.

page 160, bottom line. The sentence should read, "If x_i and x^j were independent, the dimension of $V_{k,l}$ would be . . . ." (There is a less serious correction in the third paragraph of 3.1 on page 161: "integral orbits form a discrete set . . . ")

page 168, before Proposition 5. The reference [**Kl**] should be replaced by [**Kly**].

page 168, Proposition 5. This is false as stated. For example, let $\Omega_1 = \Omega_2 = \Omega_3$ be the smallest non-trivial integral orbit for $SU(2)$. In that case $\Omega_1 + \Omega_2$ contains the integral orbits 0, Ω_1, and $2\Omega_1$. The representation π corresponding to Ω_1 is the 2-dimensional representation of $SU(2)$. The tensor product $\pi \otimes \pi$ is the sum of the 1-dimensional and the 3-dimensional representations, corresponding to the orbits 0 and $2\Omega_1$; π does not occur.

The last line of the Proposition should read, "Suppose that Ω_i is the orbit of $\lambda_i \in \mathfrak{t}^*$, and that $\lambda_1 + \lambda_2 - \lambda_3 \in Q$. Then $\Omega_1 + \Omega_2$ contains Ω_3 if and only if $\pi_1 \otimes \pi_2$ contains π_3."

page 172, Example 12. The Serre duality theorem provides a natural isomorphism

$$H^i(\mathcal{F}, \mathcal{L}_\lambda)^* \simeq H^{r-i}(\mathcal{F}, \mathcal{L}_{-\lambda-2\rho}).$$

Here r is the dimension of $\mathcal{F}$, and 2ρ is the weight of the canonical line bundle (top degree differential forms).

In the last line of Example 12(a), it is therefore more natural to say that "r-dimensional cohomology with coefficients in $\mathcal{L}_{-2\rho}$ is dual to the 0-dimensional cohomology with coefficients in the trivial sheaf $\mathcal{L}_0$.

The explanation of the last example in (b) is a little sketchy. First, the projective space at the beginning is *not* the one discussed on page 171 (of lines in $\mathbb{C}^3$) but rather the dual projective space $\widetilde{\mathbb{P}}^2(\mathbb{C})$ of two-dimensional planes in $\mathbb{C}^3$ (or lines in the dual space). With this clarification, the first assertion is that

$$H^2(\mathcal{F}, \mathcal{L}_{-4\omega_2}) \simeq H^2(\widetilde{\mathbb{P}}^2(\mathbb{C}), \mathcal{L}_{-4\omega_2}).$$

This is true. To prove it, apply the Leray spectral sequence of the projection $\mathcal{F} \to \widetilde{\mathbb{P}}^2(\mathbb{C})$ (sending a flag in $\mathbb{C}^3$ to the 2-dimensional subspace in the flag). The fiber of this projection over a plane W is the projective space $\mathbb{P}^1(\mathbb{C})$ of lines in W. The bundle is trivial on the fibers, and therefore (since the fibers are $\mathbb{P}^1$) the higher cohomology of the fibers with coefficients in the bundle is zero.

The canonical bundle for $\widetilde{\mathbb{P}}^2(\mathbb{C})$ is $3\omega_2$. Serre duality therefore guarantees that

$$H^2(\widetilde{\mathbb{P}}^2(\mathbb{C}), \mathcal{L}_{-4\omega_2}) \simeq H^0(\widetilde{\mathbb{P}}^2(\mathbb{C}), \mathcal{L}_{4\omega_2-3\omega_2})^*.$$

Arguing as at the bottom of page 171, one sees that

$$H^0(\widetilde{\mathbb{P}}^2(\mathbb{C}), \mathcal{L}_{k\omega_2}) \simeq \pi_{k\omega_2}.$$

We have therefore shown that

$$H^2(\mathcal{F}, \mathcal{L}_{-4\omega_2}) \simeq \pi^*_{\omega_2}.$$

This is indeed isomorphic to

$$\pi_{\omega_1} \simeq H^2(\mathbb{P}^2(\mathbb{C}), \mathcal{L}_{\omega_1}),$$

by a representation-theoretic calculation of the dual and the calculation from page 171. I do not know a straightforward geometric argument for the isomorphism

$$H^0(\widetilde{\mathbb{P}}^2(\mathbb{C}), \mathcal{L}_{k\omega_2})^* \simeq H^0(\mathbb{P}^2(\mathbb{C}), \mathcal{L}_{k\omega_1}).$$

page 173, Theorem 9. The reference to **[K11]** is pedagogical: to a place to learn the mathematics, rather than to an original source. As the discussion of the proof shows, this result was first proved by Harish-Chandra (announced in 1956; see page 232 of volume II of his collected works).

page 174, line 4. The reference **[K1]** is meant to be **[Ki1]**; the formula appears on page 98 of the English translation.

page 180, introduction to 1.2. For the unitary representation theory of real semisimple groups, only the reference [**Wa**] is reasonable: both [**Vo1**] and [**Zh2**] are concerned with more algebraic matters. I would add also [**Kn**] and [KV].

page 180, last paragraph. The structure theorem stated for general Lie groups is correct if G is connected and simply connected, but not in general; it is false for $U(2)$, for example.

page 184, metaplectic representation. This representation has a long and colorful history, in which I will not attempt to sort out credit. (Two of the key original references are [Sh] and [Wl].) For a student wishing to learn the mathematics, [**LV**] is a beautiful reference.

page 188, end of Example 3. The reference [**Ner**] is missing from the bibliography. Possibly [**Ner1**] is meant.

page 196, Proposition 3. The discussion of references for this result is confused. Berezin's result was published in 1957 as [Bz1]. The correction discussed in a footnote is apparently [Bz2] (published in 1963, rather than 1967).

Because of the vague statement of the result, it is difficult to be certain; but it seems likely that Proposition 3 is the result proved by Harish-Chandra in a 1956 paper (Theorem 2 on page 125 of volume II of Harish-Chandra's collected works [**H**]). This result was announced in 1955 (Lemma 2 on page 10 of volume II of the collected works).

page 220, after Proposition 2. This paragraph should be eliminated entirely. It is not possible to define relative Lie algebra cohomology groups $H^k(\mathfrak{g}, \mathfrak{h}, V)$ unless V carries a representation of $\mathfrak{g}$ (not just H). The equivariant vector bundle $E = G \times_H V$ indeed makes sense if V is only an H representation, but the space of sections of E is certainly not a local system.

If the representation V of H is trivial on the identity component H_0, then it makes sense to speak of "locally constant sections" of E. The sheaf $\mathcal{E}_V$ of germs of locally constant sections is a local system on G/H, so it makes sense to speak of the Čech cohomology of G/H with coefficients in that sheaf. It is possible to formulate a true statement vaguely related to the last sentence ("It turns out … ") but one needs to replace Lie algebra cohomology by something involving groups; and I do not know any reason to make the effort.

page 225, Proposition 3. The assertion that (the abelian group) $H_0(X, \mathbb{Z})$ is equal to (the pointed set) $\pi_0(X)$ is incorrect. What is true is that $H_0(X, \mathbb{Z})$ is the free abelian group generated by $\pi_0(X)$.

page 229, Proposition 2. This assertion is false. Take for example $G = \pm 1$ acting on X equal to the unit circle $\mathbb{T}$ (in the plane $\mathbb{R}^2$) by

$$(-1) \cdot (x, y) = (x, -y).$$

Each orbit consists of one or two points, and is therefore a closed 0-dimensional submanifold of X; but the quotient $M = X/G$ is diffeomorphic to the closed unit interval $[0, 1]$ (which is not a smooth manifold).

One might hope that the difficulty does not arise for connected groups G, but this is not the case. Take for example $G = SO(3)$ acting by conjugation on the 3-dimensional space $X = SO(3) - \{1\}$ of non-identity elements. Each orbit is a closed smooth 2-dimensional submanifold: all but one is S^2, and the remaining one is the real projective space $\mathbb{RP}^2$. The quotient manifold is diffeomorphic to the half-open interval $(0, 1]$, with the non-smooth point 1 corresponding to the projective space orbit.

page 235, Example 4. The assertion that S^{2n} admits a complex structure only for $n = 1$ has not been proved. Borel and Serre proved in [BS] that S^2 and S^6 are the only spheres admitting almost complex structures. Using the seven-dimensional representation of the compact group of type G_2, it is fairly easy to construct a G_2-invariant almost complex structure on S^6; but such a structure cannot be integrable. The non-existence of a complex structure on S^6 is a famous and long-standing open problem.

page 261, footnote. Should read, "but not positive definite, we obtain ... "

page 266, before Proposition 9. The hypotheses (a) and (b) are probably not sufficient to ensure that the coset space M_0 is a smooth manifold. I have not tried to construct an example, but the first reference implied in the text (Marsden and Weinstein [MW]) assumes also that the action of $Stab(F)$ on $\mu^{-1}(F)$ is proper.

I was not able to locate the second reference implied in the text, to work of V. I. Arnold.

page 301, before Proposition 4. The assertion that "the whole root system R can be reconstructed from the system Π of simple roots" is true only for *reduced* root systems (in which twice a root is assumed not to be a root). There are a number of similar misstatements on the following pages, like the claim after (29) (that a root system is determined by its Cartan matrix), generally corrected by adding the word "reduced."

page 304, Example 7(2). In the last line, the summation should run from 0 to 8 (not 9): "on the hyperplane $\sum_{k=0}^{k=8} x^k = 0$ form a root ... "

page 312, proof of Theorem 8. The element $\alpha^\vee$ defined here is *not* the "dual root to α." It is almost obvious that the collection $R^\vee$ of elements $\alpha^\vee$ defined here is just the image of the root system R under the isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ induced by the Killing form. In particular, this would mean that $R^\vee$ is isomorphic to R as a root system, and that is incorrect. (The dual of the root system of type B is the root system of type C.)

The dual root to α, which is generally called the *coroot* corresponding to α) is actually the element H_α defined in the next to last paragraph of page 312. The notations $\alpha^\vee$ and H_α are both widely used for the coroot.

I don't know the best way to correct this mistake, which percolates through the rest of this section and into the chapter on compact groups. Humphreys in **[Hu]** (section 8.2) uses the notation t_α for what is here written as $\alpha^\vee$. Keeping the notation $\alpha^\vee$ is *not* reasonable, since that notation is so often used for the dual root. I would therefore suggest eliminating the use of $\alpha^\vee$ altogether. The last few paragraphs of page 312 might then read as follows.

Since the Killing form is non-degenerate, we can assume (after modifying our choice of $X_{-\alpha}$ by a scalar) that $(X_\alpha, X_{-\alpha})_K = 1$. Then the element

$$t_\alpha := [X_\alpha, X_{-\alpha}]$$

has the property that

$$(t_\alpha, H)_K = \alpha(H)$$

for all $H \in \mathfrak{h}$. In other words, t_α is exactly the element of $\mathfrak{h}$ that corresponds to $\alpha \in \mathfrak{h}^*$ under the isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ induced by the Killing form.

Let $\mathfrak{g}(\alpha)$ denote the 3-dimensional subalgebra in $\mathfrak{g}$ spanned by X_α, $X_{-\alpha}$, and t_α. It is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. This algebra has a natural basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfying the commutation relations

$$[E,F] = H, \qquad [H,E] = 2E, \qquad [H,F] = -2F$$

(see Example 8 in Appendix 3.3.4 below). We can choose the isomorphism of $\mathfrak{g}(\alpha)$ with $\mathfrak{sl}(2,\mathbb{C})$ so that

$$X_\alpha \to E, \qquad 2t_\alpha/(\alpha,\alpha) \to H, \qquad 2X_{-\alpha}/(\alpha,\alpha) \to F.$$

Because of the good properties of this basis of $\mathfrak{sl}(2,\mathbb{C})$, it is convenient to define

$$H_\alpha := 2t_\alpha/(\alpha,\alpha).$$

The element H_α is sometimes called the **dual root** to α. The collection of all dual roots forms a root system in $\mathfrak{h}$. This system is denoted by $R^\vee$ and is called the **dual root system**.

From now on we will always modify $X_{-\alpha}$ by a scalar so that we have the commutation relations

$$[X_\alpha, X_{-\alpha}] = H_\alpha, \qquad [H_\alpha, X_\alpha] = 2X_\alpha, \qquad [H_\alpha, X_{-\alpha}] = -2X_{-\alpha}.$$

Let us study the adjoint action ...

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page 315, definitions of $\mathfrak{sp}(2n,\mathbb{C})$ and $\mathfrak{sp}(2n,\mathbb{R})$. Defining the Lie algebra associated to a bilinear form requires specifying the form. Here the symplectic form on column vectors v, w in $\mathbb{C}^{2n}$ or $\mathbb{R}^{2n}$ has been chosen to be

$$\omega_{2n}(v,w) = v^t J_{2n} w.$$

All that's required for this definition to give the right Lie algebra up to isomorphism is that J_{2n} be invertible and skew-symmetric. I could not locate a definition of J_{2n} in the text. I believe that most authors use

$$J'_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix};$$

this is the convention in **[OV]**, **[Hel]**, **[Hu]**, and **[Kn]**, for example. The corresponding Lie algebra has a matrix description in terms of $n \times n$ blocks as

$$\mathfrak{sp}'(2n, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid B^t = B, C^t = C, D = -A^t \right\}.$$

Here, however, it appears that the intention is to use

$$J_{2n} = \begin{pmatrix} 0 & S_n \\ -S_n & 0 \end{pmatrix}.$$

Here S_n is the "anti-diagonal" $n \times n$ matrix

$$S_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & \vdots & & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

This choice changes the description of $\mathfrak{sp}(2n, \mathbb{C})$: the requirements on the $n \times n$ blocks A, B, C, and D must now be phrased not in terms of transpose, but in terms of flipping the matrix over the anti-diagonal. For $n = 2$, for example, we get

$$\mathfrak{sp}(4, \mathbb{C}) = \left\{ \begin{pmatrix} a & b & c & d \\ e & f & g & c \\ i & j & -f & -b \\ m & i & -e & -a \end{pmatrix} \right\}.$$

This distinction matters in the matrix description of the diamond Lie algebra on page 126.

page 334, Example 2, formula (2′)**.** The limit formula for the L^∞ norm is not correct for all functions $f \in L^\infty$ without further assumptions (for example, that $\mu(X) < \infty$). If X has infinite measure, then the constant function 1 has $\|f\|_\infty = 1$, but $\|f\|_p = \infty$ for all finite p.

page 367, Remark 3. The text observes that in many examples the Gårding space (of compactly supported smooth densities applied to H) coincides with H^∞, and raises the question of whether this is always the case. Dixmier and Malliavin in [DM] prove this for any continuous Fréchet representation of a Lie group G.

page 369, Theorem 6. A footnote says that part (a) (the conjugation-invariance of generalized function characters) holds only for unimodular groups. This caveat seems to be unnecessary. For non-unimodular groups, the relationship between generalized functions and distributions is a bit more subtle (since there is no bi-invariant Haar measure). But as long as one works with the character as a generalized function (as the author is careful to do) the assertion of (a) seems to hold.

In part (c) (that the character of an irreducible representation determines the representation) there is a hypothesis that the character be non-zero. This hypothesis is automatically satisfied for unitary representations: if δ is a non-negative test density supported near the identity, then $\pi(\delta)$ will be a non-zero operator, so $\pi(\delta)\pi(\delta)^*$ will have strictly positive trace. But

$$\pi(\delta)\pi(\delta)^* = \pi(\delta * \tilde{\delta})$$

(with $\tilde{\delta}$ equal to δ twisted by the inversion map on G), so the character takes a strictly positive value at the test density $\delta * \tilde{\delta}$.

page 370, proof of Theorem 6. The reference to **[Di3]** seems to be to section 17.2.

page 371, Plancherel formula. The formulas in (13) make sense only for $f \in L^1(G) \cap L^2(G)$. The theorem should be attributed to Segal [S]. There is an account with proofs in section 18.8 of **[Di3]**.

page 376, bottom line. This should read "**Burnside formula**."

page 383, line 4. The letters M and U are interchanged in this line.

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