18.781 Problem Set 9 solutions

Due Monday April 22 in class. To answer any of the questions, you can quote theorems from the text.

1. Calculate the smallest positive solution of $x^2 - 61y^2 = -1$.

Begin with the table for calculating the continued fraction expansion of $\sqrt{61}$ from Problem Set 8; add two columns as explained below for the convergents of the continued fraction expansion. I've also added a first column with the index i.

Here is the table explained below:

i	m	q	ξ	a	h	k
0	0	1	$\sqrt{61}$	7	7	1
1	7	12	$\frac{7+\sqrt{61}}{12}$	1	8	1
2	5	3	$\frac{5+\sqrt{61}}{3}$	4	39	5
3	7	4	$\frac{7+\sqrt{61}}{4}$	3	125	16
4	5	9	$\frac{5+\sqrt{61}}{9}$	1	164	21
5	4	5	$\frac{4+\sqrt{61}}{5}$	2	453	58
6	6	5	$\frac{6+\sqrt{61}}{5}$	2	1070	137
7	4	9	$\frac{4+\sqrt{61}}{9}$	1	1523	195
8	5	4	$\frac{5+\sqrt{61}}{4}$	3	5639	722
9	7	3	$\frac{7+\sqrt{61}}{3}$	4	24079	3083
10	5	12	$\frac{5+\sqrt{61}}{12}$	1	29718	3805
11	7	1	$7+\sqrt{61}$	14	440131	56353
12	7	12	$\frac{7+\sqrt{61}}{12}$	1	469849	60158

and so on; $\sqrt{61} = \langle 7, \overline{1,4,3,1,2,2,1,3,4,1,14}\rangle$. The period $r = 11$ is odd, so Theorem 7.25 in the text guarantees that the smallest positive solution of $x^2 - 61y^2 = -1$ is $x = h_{11-1}$, $y = k_{11-1}$:

$$(x, y) = (29718, 3805)$$

A calculator will verify that $x^2 = 883,159,524$ and $61y^2 = 883,159,825$, so at least this *is* a solution.

2. Calculate the smallest positive solution of $x^2 - 61y^2 = 1$.

Theorem 7.25 says that the answer is (h_{21}, k_{21}). One way to find these is to extend the table above for an additional nine rows. This is a painful process by hand (although easy enough on a computer). A simpler solution is to use the matrix formulas from Problem 7. I won't repeat all the notation (what was (P_n, Q_n) there is what we're calling (h_n, k_n) here) but these say that

$$\begin{aligned}\begin{pmatrix} h_{21} & h_{20} \\ k_{21} & k_{20}\end{pmatrix} &= A_0A_1\cdots A_{21}\\ &= A_0A_1\cdots A_{10}A_{11}A_{12}\cdots A_{21}\\ &= A_0A_1\cdots A_{10}A_{11}A_1\cdots A_{10}\end{aligned}$$

In the last step we used the periodicity

$$A_n = A_{n+11} \qquad (n \geq 1).$$

Also

$$A_{11} = \begin{pmatrix} 14 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix} A_0.$$

Inserting this above gives

$$\begin{pmatrix} h_{21} & h_{20} \\ k_{21} & k_{20} \end{pmatrix} = A_0 \cdots A_{10} \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix} A_0 \cdots A_{10}.$$

We have a formula for $A_0 \cdots A_{10}$ from the table in #1; inserting it gives

$$\begin{aligned}
\begin{pmatrix} h_{21} & h_{20} \\ k_{21} & k_{20} \end{pmatrix} &= \begin{pmatrix} 29718 & 24079 \\ 3805 & 3083 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 29718 & 24079 \\ 3805 & 3083 \end{pmatrix} \\
&= \begin{pmatrix} 29718 & 7 \cdot 29718 + 24079 \\ 3805 & 7 \cdot 3805 + 3083 \end{pmatrix} \begin{pmatrix} 29718 & 24079 \\ 3805 & 3083 \end{pmatrix} \\
&= \begin{pmatrix} 29718 & 232105 \\ 3805 & 29718 \end{pmatrix} \begin{pmatrix} 29718 & 24079 \\ 3805 & 3083 \end{pmatrix} \\
&= \begin{pmatrix} 29718 & 61 \cdot 3805 \\ 3805 & 29718 \end{pmatrix} \begin{pmatrix} 29718 & 24079 \\ 3805 & 3083 \end{pmatrix} \\
&= \begin{pmatrix} 1766319049 & h_{20} \\ 226153980 & k_{20} \end{pmatrix};
\end{aligned}$$

I didn't do the calculations of the last two entries because we don't need them. So the smallest positive solution we are looking for is

$$x = 1766319049, \qquad y = 226153980$$

$$x^2 = 3119882982860264401, \qquad 61y^2 = 3119882982860264400.$$

There is another, even easier, way to get this. An easy generalization of Theorem 7.26 says that if (x_1, y_1) is the smallest positive solution of $x^2 - dy^2 = -1$, then all positive solutions of $x^2 - dy^2 = (-1)^n$ are the integers defined by

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n.$$

In particular, the smallest solution with $+1$ comes from $n = 2$:

$$x = x_1^2 + 61y_1^2, \qquad y = 2x_1y_1.$$

That's evidently what the matrix calculation gave (at least after I inserted the factorization of 232105 before the last step).

This is all still accessible to hand calculation. For a more serious Pell's equation, and really *big* numbers, you might Google Archimedes' cattle problem.

3. A *Pythagorean triple* consists of three positive integers x, y, and z satisfying $x^2 + y^2 = z^2$. If $a > b$ are positive integers, then

$$\text{(PT)} \qquad (a^2 - b^2, 2ab, a^2 + b^2), \qquad (2ab, a^2 - b^2, a^2 + b^2)$$

are both Pythagorean triples. A Pythagorean triple is called *primitive* if x, y, and z are relatively prime. We are going to prove in class that any primitive Pythagorean triple is given by one of the formulas (PT).

a) Find a non-primitive Pythagorean triple given by one of the formulas (PT).

Taking $a = 3$, $b = 1$ leads to the non-primitive triple $(8, 6, 10)$.

b) Find necessary and sufficient conditions on the integers $a > b > 0$ so that the triples (PT) are primitive. You should explain as completely as you can why your condtions are necessary (that is, why (PT) is *not* primitive when they fail) and why they are sufficient (that is, why (PT) *is* primitive when they hold). (Hint: one of the conditions is that a and b are relatively prime.)

The requirements are

$$a \text{ and } b \text{ are relatively prime}$$

and

$$a \text{ and } b \text{ have different parity (one even and one odd).}$$

If a and b have the common factor d, then x, y, and z have the common factor d^2, so the triple is *not* primitive. That's why the first condition is necessary. If a and b are both odd, then a^2 and b^2 are both odd, so $a^2 - b^2$ and $a^2 + b^2$ must be even. Therefore x, y, and z have the common factor 2, and the triple is *not* primitive. If a and b are both even, then they are not relatively prime, and we already know that the triple is not primitive.

Conversely, suppose these two requirements are satisfied; we want to know that the triple is primitive. Suppose that x and y have a common prime factor p. This means that $a^2 - b^2 = (a+b)(a-b)$ and $2ab$ have the common factor p. Since a and b have opposite parity, x is odd, so p must be odd. Therefore p is a factor either of a or of b, and also either of $a + b$ or $a - b$. This is four cases. For example if p is a factor of a and of $a + b$, then it must also be a factor of b, contradicting our hypothesis that a and b are relatively prime. The other three cases are identical, all leading to contradictions; so the conclusion is that x and y *cannot* have a common prime factor, as we wished to show.

c) Find an example of a non-primitive Pythagorean triple that is *not* given by one of the formulas (PT).

I'm not sure of the best systematic way to proceed. The smallest non-primitive triple is $(6, 8, 10)$. This is given by the second formula in (PT) with $a = 3$, $b = 1$. The next is $(9, 12, 15)$. If this is to be given by either formula (PT) it must be the first, since the second formula would say that 9 was even. So we want to see whether there exist $a > b > 0$ with

$$a^2 - b^2 = 9, \qquad ab = 6.$$

The only solutions to the second equation are $a = 6$, $b = 1$ and $a = 3$, $b = 2$. Neither of these satisfies the first. The conclusion is that $(9, 12, 15)$ is *not* given by a formula (PT).

d) There is a function $F\colon \mathbb{R}^2 \to \mathbb{R}^3$,

$$F(\alpha,\beta) = (\alpha^2 - \beta^2, 2\alpha\beta, \alpha^2 + \beta^2).$$

Give the simplest and most complete description you can of the image of F. (Hint: the image of F is a "parametric surface." Another example of a parametric surface is

$$G(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi),$$

spherical coordinates. An answer for G might be, "the image of G is the unit sphere $x^2 + y^2 + z^2 = 1$.")

The image in $\mathbb{R}^3$ is the half cone

$$x^2 + y^2 = z^2, \qquad z \geq 0.$$

That $F(\alpha,\beta)$ belongs to this half cone is very easy: z is a sum of squares, so must be nonnegative, and verifying the cone equation is easy algebra. Seeing that the image is the *entire* half cone requires a bit of thought. One possibility is to identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$ in the usual way; then

$$F\colon \mathbb{C} \to \mathbb{C} \times \mathbb{R}, \qquad F(w) = (w^2, |w|^2).$$

In these coordinates the equation of the cone is

$$\{(u,t) \in \mathbb{C} \times \mathbb{R} \mid t = \pm|u|\}.$$

Now it's more or less clear that the map F is two-to-one from $\mathbb{C}$ to the positive cone: the preimage of $(u, |u|)$ consists of the two square roots of the complex number u. (Well, if $u = 0$ there is only one square root.)

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Summary of the method from the text and class for calculating the continued fraction expansion of $(m_0 + \sqrt{d})/q_0$ and the convergents

$$\langle a_0, \dots, a_i \rangle = \frac{h_i}{k_i}:$$

make a table with rows numbered $i = 0, 1, 2, \dots$, and six columns of data: m_i, q_i, $\xi_i = (m_i + \sqrt{d})/q_i$, $a_i = [\xi_i]$, h_i, and k_i. Calculate row $i+1$ from row i by the formulas

$$m_{i+1} = q_i a_i - m_i, \qquad q_{i+1} = (d - m_{i+1}^2)/q_i.$$

This works as long as m_0 is an integer, d is a positive integer non-square, and q_0 is a divisor of $d - m_0^2$.

For the convergents: $h_i = a_i h_{i-1} + h_{i-2}$, $k_i = a_i k_{i-1} + k_{i-2}$. These formulas get started with $h_{-2} = 0$, $h_{-1} = 1$, $k_{-2} = 1$, $k_{-1} = 0$.