18.781 Problem Set 9

Due Monday April 22 in class. To answer any of the questions, you can quote theorems from the text.

1. Calculate the smallest positive solution of $x^2 - 61y^2 = -1$.

2. Calculate the smallest positive solution of $x^2 - 61y^2 = 1$.

3. A *Pythagorean triple* consists of three positive integers x, y, and z satisfying $x^2 + y^2 = z^2$. If $a > b$ are positive integers, then

$$\text{(PT)} \qquad (a^2 - b^2, 2ab, a^2 + b^2), \qquad (2ab, a^2 - b^2, a^2 + b^2)$$

are both Pythagorean triples. A Pythagorean triple is called *primitive* if x, y, and z are relatively prime. We are going to prove in class that any primitive Pythagorean triple is given by one of the formulas (PT).

a) Find a non-primitive Pythagorean triple given by one of the formulas (PT).

b) Find necessary and sufficient conditions on the integers $a > b > 0$ so that the triples (PT) are primitive. You should explain as completely as you can why your condtions are necessary (that is, why (PT) is *not* primitive when they fail) and why they are sufficient (that is, why (PT) *is* primitive when they hold). (**Hint**: one of the conditions is that a and b are relatively prime.)

c) Find an example of a non-primitive Pythagorean triple that is *not* given by one of the formulas (PT).

d) There is a function $F\colon \mathbb{R}^2 \to \mathbb{R}^3$,

$$F(\alpha, \beta) = (\alpha^2 - \beta^2, 2\alpha\beta, \alpha^2 + \beta^2).$$

Give the simplest and most complete description you can of the image of F. (**Hint**: the image of F is a "parametric surface." Another example of a parametric surface is

$$G(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi),$$

spherical coordinates. An answer for G might be, "the image of G is the unit sphere $x^2 + y^2 + z^2 = 1$.")

****

Summary of the method from the text and class for calculating the continued fraction expansion of $(m_0 + \sqrt{d})/q_0$ and the convergents

$$\langle a_0, \dots, a_i \rangle = \frac{h_i}{k_i}:$$

make a table with rows numbered $i = 0, 1, 2, \dots$, and six columns of data: m_i, q_i, $\xi_i = (m_i + \sqrt{d})/q_i$, $a_i = [\xi_i]$, h_i, and k_i. Calculate row $i+1$ from row i by the formulas

$$m_{i+1} = q_i a_i - m_i, \qquad q_{i+1} = (d - m_{i+1}^2)/q_i.$$

This works as long as m_0 is an integer, d is a positive integer non-square, and q_0 is a divisor of $d - m_0^2$.

For the convergents: $h_i = a_i h_{i-1} + h_{i-2}$, $k_i = a_i k_{i-1} + k_{i-2}$. These formulas get started with $h_{-2} = 0$, $h_{-1} = 1$, $k_{-2} = 1$, $k_{-1} = 0$.