18.781 Problem Set 8 solutions

1. Prove that for any positive integer m, $\langle\overline{2m}\rangle = m + \sqrt{m^2+1}$.

The value $x = \overline{2m}\rangle$ by definition satisfies the equation

$$x = 2m + \frac{1}{x}, \qquad x^2 = 2mx + 1, \qquad x^2 - 2mx - 1 = 0.$$

The roots of this quadratic are $(-2m \pm \sqrt{4m^2+4})/2 = -m \pm \sqrt{m^2+1}$. Because the continued fraction is clearly non-negative, the root is the one with $+$.

2. Write down *all* the quadratic irrationals whose continued fraction is periodic of period 1. (Problem 1 writes down some of them.)

We solve for $x = \overline{a}\rangle$ (with a a positive integer) in exactly the same way, getting the solution

$$= \frac{a + \sqrt{a^2+4}}{2}.$$

3. Write down *all* the quadratic irrationals whose continued fraction is periodic of period 2.

The defining equation for $x = \langle\overline{a,b}\rangle$ (with a and b positive integers) is

$$\begin{aligned} x &= a + \frac{1}{b + \frac{1}{x}} = a + \frac{1}{\frac{bx+1}{x}} \\ &= a + \frac{x}{bx+1} = \frac{abx + a + x}{bx+1}. \end{aligned}$$

Clearing denominators leads to the quadratic equation

$$bx^2 - abx - a = 0, \qquad x = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2b}.$$

(We need to use the positive square root since the square root is strictly larger than ab, so the other root is negative.)

It's fine to stop here, but here are a few more words. You might want a formula for x that makes it relatively easy to see by inspection whether a given quadratic irrational has this form. If we use $c = ab$, then such a formula is

$$x = \frac{1 + \sqrt{1 + 4/c}}{2} \cdot d \qquad (c \geq d > 0 \text{ integers}, \ c|d).$$

4. Suppose that a and b are strictly positive integers. Explain why $a^2b^2 + 4ab$ is *not* a perfect square. (It's OK to say "this follows from Problem 3" if you add a couple of additional words of explanation. But it's also possible to solve this problem independently.)

Problem 3 says that $\sqrt{a^2b^2+4ab}$ is the irrational part of an infinite continued fraction; so it cannot be rational, so this number cannot be a perfect square.

More directly, writing $N = ab$ (a positive integer), the number in question is $(N+2)^2 - 4$. Its square root is therefore strictly smaller than $N+2$. On the other hand,

$$(N^2 + 4N) - (N+1)^2 = 2N - 1 > 0,$$

so also its square root is strictly larger than $N+1$. Since there are no integers between $N+1$ and $N+2$, our number cannot be a perfect square.

5. Calculate the continued fraction expansion of $\sqrt{61}$.

Here is the table explained below:

m	q	ξ	a
0	1	$\sqrt{61}$	7
7	12	$\frac{7+\sqrt{61}}{12}$	1
5	3	$\frac{5+\sqrt{61}}{3}$	4
7	4	$\frac{7+\sqrt{61}}{4}$	3
5	9	$\frac{5+\sqrt{61}}{9}$	1
4	5	$\frac{4+\sqrt{61}}{5}$	2
6	5	$\frac{6+\sqrt{61}}{5}$	2
4	9	$\frac{4+\sqrt{61}}{9}$	1
5	4	$\frac{5+\sqrt{61}}{4}$	3
7	3	$\frac{7+\sqrt{61}}{3}$	4
5	12	$\frac{5+\sqrt{61}}{12}$	1
7	1	$7+\sqrt{61}$	14
7	12	$\frac{7+\sqrt{61}}{12}$	1
5	3	$\frac{5+\sqrt{61}}{3}$	4

and so on; $\sqrt{61} = \langle 7, \overline{1,4,3,1,2,2,1,3,4,1,14} \rangle$.

Summary of the method from the text and class for calculating the expansion of $(m_0+\sqrt{d})/q_0$: make a table with rows numbered $i = 0,1,2,\dots$, and four columns of data: m_i, q_i, $\xi_i = (m_i+\sqrt{d})/q_i$, and $a_i = [\xi_i]$. Calculate row $i+1$ from row i by the formulas

$$m_{i+1} = q_i a_i - m_i, \qquad q_{i+1} = (d - m_{i+1}^2)/q_i.$$

This works as long as m_0 is an integer, d is a positive integer non-square, and q_0 is a positive divisor of $d - m_0^2$. Then (for all $i \geq 1$) $d - m_i^2 > 0$, and q_i is a positive divisor of $d - m_i^2$.