18.781 Problem Set 6

Due Monday, April 8 in class.

This problem set is about continued fractions. To fix the notation, I'll write here a little of what's written in the text. The starting point is two integers

$$u_0, u_1, \qquad u_1 \geq 1.$$

The algorithm for computing the continued fraction expansion is very much like the Euclidean algorithm: repeated division with remainder

$$\begin{aligned} u_0 &= u_1 a_0 + u_2, && (0 \leq u_2 < u_1) \\ u_1 &= u_2 a_1 + u_3, && (0 \leq u_3 < u_2) \\ &\vdots \\ u_{n-1} &= u_n a_{n-1} + u_{n+1} && (0 \leq u_{n+1} < u_n) \\ u_n &= u_{n+1} a_n. \end{aligned}$$

Then

$$\begin{aligned} \frac{u_0}{u_1} &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_{n-1} + \cfrac{1}{a_n}}}} \\ &=_{\text{def}} \langle a_0, \ldots, a_n \rangle. \end{aligned}$$

1a. The text says that you should start with a fraction u_0/u_1 in lowest terms; that is, with the property that u_0 and u_1 have no common factor. If you do that, what is the value of u_{n+1}?

1b. Explain what happens in the algorithm above if you start with a fraction u_0/u_1 that is **not** in lowest terms.

2. Define

$$A_j = \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix},$$

and define

$$\begin{pmatrix} P_j & P_{j-1} \\ Q_j & Q_{j-1} \end{pmatrix} = A_0 A_1 \cdots A_j. \qquad (n \geq j \geq 0$$

Prove that for all $0 \leq j \leq n$

$$\langle a_0, \ldots, a_j \rangle = P_j/Q_j,$$

$$P_j Q_{j-1} - P_{j-1} Q_j = (-1)^{j+1},$$

$$Q_0 = 1, \qquad Q_{j+1} > Q_j,$$

$$\frac{P_j}{Q_j} - \frac{P_{j-1}}{Q_{j-1}} = \frac{(-1)^{j+1}}{Q_j Q_{j-1}}.$$

3. If $x > 0$ is any real number, define

$$\langle a_0, \ldots, a_n, x \rangle =_{\text{def}}= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_{n-1} + \cfrac{1}{a_n + \cfrac{1}{x}}}}}$$

(If x is a positive integer, this is consistent with our notation for continued fractions.) Using the notation of Problem 2, prove that

$$\langle a_0, \ldots, a_n, x \rangle = \frac{P_n x + P_{n-1}}{Q_n x + Q_{n-1}}.$$

4. Find an explicit formula (something like $4 - 2\sqrt{3}$) for the periodic continued fraction

$$\langle 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots \rangle = \langle \overline{1, 2, 3} \rangle.$$

(Hint: if you use the previous problems, you can make most of the arithmetic into multiplying some 2×2 matrices.)