18.781 Problem Set 6

Due Monday, October 24 in class.

1. Suppose that

\[ Q(x, y) = Ax^2 + Bxy + Cy^2 \]

is a binary integral quadratic form. Recall from class that the discriminant of \( Q \) is defined to be

\[ D = B^2 - 4AC. \]

The form \( Q \) is said to represent zero if there are integers \( x \) and \( y \), not both zero, such that \( Q(x, y) = 0 \).

1(a). Suppose that either \( A \) or \( C \) is zero. Show that then \( Q \) represents zero, and the discriminant is a perfect square.

1(b). Suppose that \( A \) and \( C \) are both non-zero, \( (x, y) \neq (0, 0) \), but \( Q(x, y) = 0 \). Explain why \( x \) and \( y \) are both non-zero.

1(c). Suppose that \( A \) and \( C \) are both non-zero, \( (x, y) \neq (0, 0) \), but \( Q(x, y) = 0 \). Prove that \((2Ax + By)/y\) is a (rational) square root of \( D \). Explain why it follows that \( D \) must be a perfect square.

1(d). Suppose that \( A \) and \( C \) are both non-zero, and that \( D \) is a perfect square. Prove that \( Q \) represents zero.

1(e). Suppose that \( Q \) represents zero. Prove that there is a new coordinate system

\[ u = px + qy, \quad v = rx + sy \]

(with \( p, q, r, s \) integers satisfying \( ps - qr = 1 \)) so that in the new coordinates

\[ Q(u, v) = A'u^2 + B'uv, \]

with \( B' \) a square root of \( D \). (Hint: you can quote results from the exercises for section 5.7. This problem is still a bit more than I should really expect you to be able to do, so don’t lose sleep over it.)

2. I proved in class Dirichlet’s theorem that if \( \xi \) is any irrational number, then there are infinitely many rational numbers \( p/q \) so that

\[ |\xi - p/q| < 1/q^2. \]

The point of this problem is to show that you can’t do much better than this in general. Define

\[ \xi_0 = (1 + \sqrt{5})/2, \]

the larger root of the equation

\[ f(x) = x^2 - x - 1. \]

This is the Golden Ratio, about which you can read in art history classes as well as in mathematics. I’ll talk about this equation using the binary quadratic form

\[ Q(p, q) = p^2 - pq - q^2. \]
2(a). Calculate the discriminant of the quadratic form $Q$. Explain why $Q$ is indefinite and does not represent zero.

2(b). List the pairs of numbers appearing on successive edges of John Conway’s river for the quadratic form $Q$. (The sequence of pairs eventually repeats; you can list the pairs in one period, then put a bar over it.)

2(c). Show that if $p$ and $q$ are any integers with $q$ not zero, then

$$|f(p/q)| \geq 1/q^2.$$  

2(d). Suppose that

$$1/2 \leq x \leq \xi_0.$$  

Prove that

$$|f(x)| \leq (\xi_0 - x)(2\xi_0 - 1) = (\xi_0 - x)(\sqrt{5}).$$  

Deduce that

$$|\xi_0 - x| \geq f(x)/\sqrt{5}.$$  

2(e). Suppose that $p/q$ is a rational approximation to $\xi_0$, and that $p/q < \xi_0$. Prove that

$$|\xi_0 - p/q| > 1/\sqrt{5}q^2.$$  

**Remark.** This problem says that $\xi_0$ can’t be very well approximated by rational numbers from below; for example, you can’t get something like Dirichlet’s theorem with $1/3q^2$ in place of $1/q^2$. (It’s also true that you can’t get better-than-1/$\sqrt{5}q^2$ approximations from above, but the inequalities are not quite so simple as in 2(d) above.) The number $1/\sqrt{5}$ is best possible: a theorem of Hurwitz says that if $\xi$ is any irrational, then there are infinitely many rationals $p/q$ with

$$|\xi - p/q| < 1/\sqrt{5}q^2.$$  

Do you see why this doesn’t contradict 2(e)?

3. Write down a specific irrational number $\xi_1$ with the property that for every positive integer $k$, there are infinitely many rational numbers $p/q$ such that

$$|\xi_1 - p/q| < 1/q^k.$$  

(Hint: taking a finite number of terms in the decimal expansion of $\xi$ gives a rational approximation to $\xi$. Usually the error in this approximation is terrible (like $1/q$), but sometimes it’s much smaller.)