18.781 Problem Set 5

Due Monday, October 17 in class.

1. You might remember from calculus Newton’s method for finding roots of the equation \( f(t) = 0 \). The idea is to begin with an approximation \( t_0 \) to a root; to write down the linear approximation

\[
 f(t) \approx f(t_0) + f'(t_0)(t - t_0),
\]

and then to choose as a (hopefully better) approximation to a root of \( f \) a root \( t_1 \) of the linear equation

\[
 0 = f(t_0) + f'(t_0)(t_1 - t_0);
\]

that is, to define

\[
 t_1 = t_0 - \frac{f(t_0)}{f'(t_0)}.
\]

Now you can repeat the process starting with \( t_1 \) in place of \( t_0 \). This makes sense as long as \( f'(t_1) \neq 0 \), and under favorable conditions the sequence \( \{t_i\} \) converges to a root of \( f \). In this problem I’ll look at the function \( f(t) = t^2 - N \), with \( N \) a positive integer; approximating roots of \( f \) means approximating \( \sqrt{N} \).

1(a). Show that if \( t_0 \) is any positive number (regarded as an approximation to \( \sqrt{N} \)) then Newton’s method leads to the new approximation

\[
 t_1 = \frac{1}{2}(t_0 + \frac{N}{t_0}).
\]

(You can think of this as saying that if \( t_0 \) is (for instance) a little smaller than \( \sqrt{N} \), then \( N/t_0 \) is roughly the same amount larger than \( \sqrt{N} \), so the average of these two numbers is quite close to \( \sqrt{N} \).)

1(b). Show that if \( x_0 \) and \( y_0 \) are positive integers, and we think of \( t_0 = x_0/y_0 \) as a rational approximation to \( \sqrt{N} \), then Newton’s method leads to a new rational approximation

\[
 t_1 = x_1/y_1, \quad x_1 = x_0^2 + Ny_0^2, \quad y_1 = 2x_0y_0.
\]

1(c). Suppose that \((x_0, y_0)\) is a solution of Pell’s equation \( x^2 - Ny^2 = 1 \). Prove that the pair \((x_1, y_1)\) defined in (b) is also a solution.

1(d). You may notice that the new solutions to Pell’s equation provided by (c) all have \( y \) even. Suppose that \( N \) is an even integer not divisible by 4. Prove that if \((x, y)\) is any solution of Pell’s equation \( x^2 - Ny^2 = 1 \), then \( y \) must be even.

2. I’ll be proving in class that if \( \xi \) is any irrational real number, then there are infinitely many rational numbers \( p/q \) such that

\[
 |\xi - p/q| < 1/q^2.
\]

Prove that this assertion is false for every rational number: that is, that if \( r \) is a rational number, then there are only finitely many distinct rational numbers \( p/q \) such that

\[
 |r - p/q| < 1/q^2.
\]

3. Problem 5.4.5 in the text on page 83.