18.781 Problem Set 1 solutions

1. There is no really good notation for writing numbers in arbitrary bases, because of the need to make up more and more distinct symbols. For this problem, I'll write a number in base 31 as

$$(15)(6)(25)_{31} = 15 \cdot 31^2 + 6 \cdot 31 + 25 = 14426;$$

here all the numbers on the right, and in parentheses on the left, are base 10. When the base is ten or less, so that each parenthetical term on the left is a single digit, I'll omit the parentheses.

1(a). Write 2019 **in base** 17**.**

The algorithm explained in class is to repeatedly divide the number by 17; the sequence of remainders (read right to left) is the base 17 representation. We find

$$\begin{aligned} 2019/17 &= 118 \quad R\ 13 \\ 118/17 &= 6 \quad R\ 16 \\ 6/17 &= 0 \quad R\ 6 \end{aligned}$$

The conclusion is

$$2019_{10} = (6)(16)(13)_{17}.$$

1(b). What was the most recent year divisible by 17**?**

If the question were "most recent year divisible by 10, you'd just make the last digit zero and answer 2010**,** 9 **years ago**. So you drop the last base 17 digit and answer $(6)(16)(0)_{17}$**,** 13 **years ago; so** 2006_{10}.

1(c). What was the most recent year divisible by 289?

Drop the last two base 17 digits: $(6)(0)(0)_{17}$**,** $(16)(13)_{17} = 285$ **years ago; so** 1734_{10}.

2. Division with remainder works for dividing by *any* positive integer, including 1**. Explain why this does not lead to a "base** 1**" representation of any positive integer as a string of zeros.**

The algorithm for computing the base a representation continues until you get a quotient of zero; then repeated division keeps giving 0 with a remainder of zero, so you have all the base a digits. If you divide N by 1, you get N with a remainder of 0; so if $N \neq 0$ the algorithm never comes to an end; it produces an infinite string of 0s whatever N is. This seems not very useful; in any case it is *not* producing something like "a string of N 0s."

Another way to state the difficulty is that the base b representation writes N as a sum of powers of b, with coefficients between 0 and $b-1$. So a base 1 representation tries to write N as a sum of powers of 1, *with all coefficients equal to zero.* This is clearly not possible for nonzero N.

3. An integer written in base 10 is divisible by 3 if and only if the sum of the digits is divisible by 3.

3(a). Explain why this is true.

The base 10 representation of N means

$$N = a_0 + a_1 \cdot 10^1 + a_2 \cdot 10^2 + \cdots + a_m \cdot 10^m.$$

Look at this equation modulo 3; that is, as in equation for N in $\mathbb{Z}/3\mathbb{Z}$. Since 10 is equal to 1 modulo 3, every power of 10 is 1 modulo 3. The equation is therefore

$$N \cong a_0 + a_1 \cdot 1 + \cdots + a_m \cdot 1 \pmod{3}.$$

That is, N and the sum of its digits have the same remainder when divided by 3. In particular, N is divisible by 3 if and only if the sum of the digits is divisible by 3.

3(b). Give a similar method to decide whether an integer written in base 8 is divisible by 3.

This time the base 8 is -1 modulo 3; so the even powers of 8 are $+1$ modulo 3 and the odd powers are -1. Therefore

$$N = b_0 + b_1 \cdot 8^1 + b_2 \cdot 8^2 + \cdots + b_n \cdot 8^n$$

implies that

$$N \cong a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^m a_m \pmod{3}.$$

That is, N **is divisible by** 3 **if and only if the** *alternating* **sum of its base** 8 **digits is divisible by** 3**.**

3(c). What's the remainder when $(25)(11)(7)(17)_{31}$ is divided by 30?

The base 31 is congruent to 1 modulo 30, so every power of 31 is as well. Thererefore

$$N \cong \text{sum of its base 31 digits} \pmod{30}.$$

Because

$$25 + 11 + 7 + 17 = 60$$

is divisible by 30, this is true of $(25)(11)(7)(17)_{31}$ as well: the **remainder is zero**.

4. It's often said that zero is critical to decimal notation, because it keeps track of what power of ten each digit represents. Like many things that everyone knows, this is nonsense. There is is a version of the division algorithm in which the remainders are taken not between 0 **and** $a-1$ **but between** 1 **and** a**. Using this algorithm, one finds that every positive integer** N **has a unique expression**

$$N = R_m \cdot 8^m + \cdots + R_1 \cdot 8 + R_0,$$

with $1 \le R_i \le 8$**. We call this the** *super 8 expression*, **and write it**

$$(R_m)(R_{m-1}) \cdots (R_1)(R_0)^8 = N.$$

For example,

$$368^8 = 370_8, \qquad 588^8 = 610_8.$$

4(a). Find the super 9 expression for $738_{10} = 1010_9$.

Proceed just as for base 9, except dividing with remainders from 1 to 9. We get

$$\begin{aligned} 738/9 &= 81 \quad R\ 9 \\ 81/9 &= 8 \quad R\ 9 \\ 8/9 &= 0 \quad R\ 8 \end{aligned}$$

Therefore

$$739_{10} = 899^9.$$

4(b). What does this problem have to do with problem 2?

If you apply division-by-1-with-remainder-1 for example to 5, you get

$$\begin{aligned} 5/1 &= 4 \quad R\ 1 \\ 4/1 &= 3 \quad R\ 1 \\ 3/1 &= 2 \quad R\ 1 \\ 2/1 &= 1 \quad R\ 1 \\ 1/1 &= 0 \quad R\ 1. \end{aligned}$$

Therefore the super 1 expression of 5 is

$$5_{10} = 11111^1.$$

Any positive integer N is has super 1 expression a string of N 1s. This is not brilliant, but it works; base 1 representation fails, but super 1 representation works. (This is the system of scratching one line for each day, so it's even *useful* for desert islands and similar settings.)