Representations of Lie Groups 18.758

This will be an introduction to the infinite-dimensional unitary representations of reductive Lie groups. I'll emphasize representation theory as a non-commutative deformation of module theory. The central tool is the enveloping algebra $U(\mathfrak{g})$ of a complex reductive Lie algebra $\mathfrak{g}$; this is a non-commutative deformation of the polynomial ring $S(\mathfrak{g})$.

I will begin with Harish-Chandra's dictionary relating group representations and $(\mathfrak{g}, K)$ modules; since the latter are modules for $U(\mathfrak{g})$, this provides the non-commutative side of the picture. As a reality check for the flights of fancy to come, I'll do Bargmann's classification of the irreducible unitary representations of $SL(2, \mathbb{R})$.

If X is any finitely generated $U(\mathfrak{g})$ module, then we can attach to X a finitely generated $S(\mathfrak{g})$ module $\operatorname{gr} X$. This passage from non-commutative to commutative modules is (more or less) an analogue of passage to the classical limit in quantum mechanics. I'll look at three basic questions about this classical limit:

1) what commutative objects can appear as classical limits of $(\mathfrak{g}, K)$ modules?
2) what $(\mathfrak{g}, K)$ modules can have the same classical limits?
3) how can we reverse the classical limit process to find a "quantization" procedure constructing representations out of commutative objects?

These questions turn out to be versions of the Kirillov-Kostant orbit method, which will be a recurring mirage in the deserts we'll traverse.

In order to give even partial answers to these questions I will discuss known constructions of unitary representations (by various kinds of parabolic induction) and the (closely related) Langlands classification of irreducible $(\mathfrak{g}, K)$ modules.

There will be no required text. A good reference for the early material (and much else) is Knapp's book *Representations of semisimple Lie groups: an overview based on examples*. For the past two years I have (not) been writing a paper concerned with questions (1) and (2) above; no doubt it will be finished in the course of the semester. I'll distribute copies together with tickets to the Red Sox home games for the 2001 World Series.

The most important background needed is the representation theory of compact groups. Necessary structure theory for Lie groups and Lie algebras will be explained (mostly without proofs) as needed.

Time: MWF 2:00-3:00

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