Representations of Lie Groups 18.758

Tuesday/Thursday 9:30–11 E17-129

The course is about irreducible unitary representations of a real reductive Lie group G. I will describe (not necessarily in this or any other order) an algorithm to classify irreducible unitary representations; the mathematics needed to formulate this algorithm; software to carry out this algorithm and other calculations related to reductive groups; and interesting open questions in these directions.

Here is the setting. A *representation* of G is an action of G by linear operators on a complex topological vector space V. It is *irreducible* if V has precisely two G-invariant closed subspaces. It is *Hermitian* if V is equipped with a non-degenerate Hermitian form preserved by G. If V is irreducible, then the Hermitian form is unique up to a non-zero real scalar multiple if it exists. The representation is *unitary* if the form is positive definite.

The classification of irreducible Hermitian representations, due to Langlands, Knapp, and Zuckerman, has been known for about thirty years. The central idea is quite easy: it is analogous to the classification of highest weight modules by the highest weight. Details appear in the recommended text *Representation theory of semisimple groups* by Knapp.

A n-dimensional vector space with a non-degenerate Hermitian form has a *signature* (p, q), with p the largest dimension of a positive-definite subspace, and q the largest dimension of a negative-definite subspace. Necessarily $p + q = n$.

A Hermitian irreducible representation V of G is usually infinite-dimensional, but has a natural orthogonal decomposition into finite-dimensional subspaces $V(\delta)$; so one can define a signature $(p_V(\delta), q_V(\delta))$ (one subspace at a time) for V. Replacing the form on V by its negative replaces the signature by (q_V, p_V). A Hermitian representation V is unitary if and only if $q_V = 0$.

To classify unitary representations of G, it suffices to calculate all p_V and q_V.

The goal of this course is to describe an algorithm for making this calculation; it is the subject of a paper `arXiv:1212.2192` with Jeffrey Adams, Marc van Leeuwen, and Peter Trapa.

The algorithm is in three steps. The first step concerns a new kind of Hermitian forms on irreducible representations called "c-invariant Hermitian forms." We find a formula relating the signatures of c-invariant forms on irreducible representations to those of c-invariant forms on *standard representations* (analogues of Verma modules). This step uses certain generalizations of Kazhdan-Lusztig polynomials defined for symmetric spaces by Lusztig *et al.* in 1983.

The second step relates the signatures of c-invariant forms to those of ordinary invariant forms. This step uses some further generalizations of KL polynomials, related to the action of an outer automorphism, defined by Lusztig *et al.* in `arXiv:1206.0634`.

The third step calculates the signatures of invariant forms on standard representations, by counting sign changes in appropriate analytic functions using the orders of their zeros. The hard part is calculating the orders of zeros; this was done by Beilinson and Bernstein in a 1993 paper.

There is software `http://atlas.math.umd.edu/software/` that understands the structure theory of real reductive groups and the Langlands classification, and can carry out the signature algorithm in most cases. I hope you can learn to use the software as the course progresses.

The most important background for this course is the representation theory of compact Lie groups. A nice reference is Chapters IV and V of Knapp's book *Lie Groups Beyond an Introduction*, but there are many others. Necessary structure theory for Lie groups and Lie algebras will be explained (often without proofs) as needed.

There is no required text. The recommended texts are *Cohomological Induction and Unitary Representations* (Knapp-Vogan), and *Representation Theory of Semisimple Groups* (Knapp). You can learn a great deal from either, but you shouldn't need to look at them to follow the course.

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