Representations of Lie Groups 18.758

The course is about irreducible unitary representations of a real reductive Lie group G. I want to describe an algorithm to classify these representations.

Here is the setting. A *representation* of G is an action of G by linear operators on a complex topological vector space V. It is *irreducible* if V has precisely two G-invariant closed subspaces. It is *Hermitian* if V is equipped with a non-degenerate Hermitian form preserved by G. If V is irreducible, then the Hermitian form is unique up to a non-zero real scalar multiple if it exists. The representation is *unitary* if the form is positive definite.

The classification of irreducible Hermitian representations, due to Langlands, Knapp, and Zuckerman, has been known for about thirty years. The central idea is quite easy: it is analogous to the classification of highest weight modules by the highest weight.

A n-dimensional vector space with a non-degenerate Hermitian form has a *signature* (p, q), with p the largest dimension of a positive-definite subspace. Necessarily $p + q = n$.

A Hermitian irreducible representation V of G is usually infinite-dimensional, but has a natural decomposition into finite-dimensional subspaces; so one can define a signature (p_V, q_V) for V. Replacing the form on V by its negative replaces the signature by (q_V, p_V). A Hermitian representation V is unitary if and only if $q_V = 0$.

To classify unitary representations of G, it suffices to calculate all p_V and q_V.

That is the goal of this course; it is the subject of work in progress with Jeffrey Adams, Marc van Leeuwen, Peter Trapa, and Wai Ling Yee. I hope to present a complete algorithm for calculating p_V and q_V, and an incomplete proof that the algorithm is correct.

The algorithm is in three steps. The first step concerns a new kind of Hermitian forms on irreducible representations called "c-invariant Hermitian forms." We find a formula relating the signatures of c-invariant forms on irreducible representations to those of c-invariant forms on *standard representations* (analogues of Verma modules). This step uses certain generalizations of Kazhdan-Lusztig polynomials defined for symmetric spaces by Lusztig *et al.* in 1983.

The second step relates the signatures of c-invariant forms to those of ordinary invariant forms. This step uses some further generalizations of KL polynomials, related to the action of an outer automorphism, that are now being defined by Lusztig *et al.*

The third step calculates the signatures of invariant forms on standard representations, by counting sign changes in appropriate analytic functions using the orders of their zeros. The hard part is calculating the orders of zeros; this was done by Beilinson and Bernstein in a 1993 paper.

One of the simplest special cases for the new KL polynomials has to do with a group $G = G_1 \times G_1$, and the outer automorphism exchanging the two factors. In that case the 1983 polynomials (indexed by a pair of elements of the Weyl group W of G_1 are the original KL polynomials, and the new polynomials (indexed by a pair of involutions in W) are... new. They are related to the old problem of defining an "involution model" of the representations of W.

There is software that understands the structure theory of real reductive groups and the Langlands classification: `http://atlas.math.umd.edu/software/`. I hope you can learn to use the software as the course progresses.

The most important background for this course is the representation theory of compact Lie groups. A nice reference is Chapters IV and V of Knapp's book *Lie Groups Beyond an Introduction*, but there are many others. Necessary structure theory for Lie groups and Lie algebras will be explained (often without proofs) as needed.

There is no required text. The recommended texts are *Cohomological Induction and Unitary Representations* (Knapp-Vogan), and *Representation Theory of Semisimple Groups* (Knapp). You can learn a great deal from either, but you shouldn't need to look at them to follow the course.

MWF 1:00-2:00, Room 2-151

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