

Representations of Lie Groups 18.758

The course is about irreducible unitary representations of a real reductive Lie group G . I want to describe a conjecture for an algorithm to classify these representations.

Here is the setting. (I'll omit topological niceties, filling these in near the beginning of the course.) A *representation* of G is an action of G by linear operators on a complex vector space V . It is *irreducible* if V has precisely two G -invariant subspaces. It is *Hermitian* if V is equipped with a non-degenerate Hermitian form preserved by G . If V is irreducible, then the Hermitian form is unique up to a non-zero real scalar multiple if it exists. The representation is *unitary* if the form is positive definite.

The classification of irreducible Hermitian representations, due to Langlands, Knapp, and Zuckerman, has been known for about thirty years. The central idea is quite easy: it is analogous to the classification of highest weight modules by the highest weight.

A n -dimensional vector space with a non-degenerate Hermitian form has a well-defined *signature* (p, q) , with p the largest dimension of a positive-definite subspace and q the largest dimension of a negative-definite subspace. Necessarily $p + q = n$, and the signature characterizes the form up to isomorphism.

A Hermitian irreducible representation V of G is usually infinite-dimensional, but has a natural decomposition into finite-dimensional subspaces; so one can define a signature (p_V, q_V) for V . (To find out what p_V and q_V are, you need to take the course.) Replacing the form on V by its negative replaces the signature by (q_V, p_V) .

A Hermitian representation V is unitary if and only if $q_V = 0$.

To classify unitary representations of G , it suffices to calculate all p_V and q_V .

That is the goal of this course; it is the subject of work in progress with Jeffrey Adams, Marc van Leeuwen, Peter Trapa, and Wai Ling Yee. I hope to present a complete algorithm for calculating p_V and q_V , and an incomplete proof that the algorithm is correct.

The algorithm is in two steps. The first step relates Hermitian forms on irreducible representations to Hermitian forms on *standard representations* (analogues of Verma modules), using a generalization of Kazhdan-Lusztig polynomials. The second step calculates the signatures of Hermitian forms on standard representations.

There is software that understands the structure theory of real reductive groups and the Langlands classification: <http://atlas.math.umd.edu/software/>. I hope you can learn to use the software as the course progresses.

The most important background for this course is the representation theory of compact Lie groups. A nice reference is Chapters IV and V of Knapp's book *Lie Groups Beyond an Introduction*, but there are many others. Necessary structure theory for Lie groups and Lie algebras will be explained (often without proofs) as needed.

There is no required text. The recommended texts are *Cohomological Induction and Unitary Representations* (Knapp-Vogan), and *Representation Theory of Semisimple Groups* (Knapp). You can learn a great deal from either, but you shouldn't need to look at them to follow the course.

MWF 1:00-2:00, Room 2-255

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