

## Representations of Lie Groups 18.758

The goal of this course is to explain some of the ideas in the “local Langlands conjectures.” Since the first word in this phrase is “local,” I should begin by saying what “global” means. A global field is a finite extension  $k$  of the rational numbers (except when it isn’t; but this is only an advertisement, so you can’t expect me to tell you *everything*). The Langlands conjectures relate two kinds of number theory for  $k$ . On one side is analytic number theory, including things like automorphic forms. On the other side is algebraic number theory, including things like algebraic extensions of  $k$ . Of course such field extensions are entirely controlled by the Galois group  $\Gamma = \text{Gal}(\bar{k}/k)$ . The global Langlands conjectures assert (among many other things) that there is a correspondence

$n$ -dimensional representations  $\tau$  of  $\Gamma \rightarrow$  automorphic forms on  $GL(n)$ .

To say more, I need to be a bit more precise. The field  $k$  carries a collection of arithmetically interesting absolute values  $|\cdot|_v$ . These absolute values are indexed by “places”  $v$  of  $k$ . (If  $k = \mathbb{Q}$ , the places are all the prime numbers and  $\infty$ ; the absolute value  $|\cdot|_\infty$  is the usual one.) The completion  $k_v$  of  $k$  (with respect to the metric defined by the absolute value  $|\cdot|_v$ ) is a locally compact field. (For example,  $\mathbb{Q}_\infty = \mathbb{R}$ , and  $\mathbb{Q}_p$  is the field of  $p$ -adic numbers.) The group  $GL(n, k_v)$  of  $n \times n$  invertible matrices with entries in  $k_v$  is therefore a locally compact topological group. An automorphic form for  $k$  “is” a collection of irreducible representations  $(\pi_v, \mathcal{H}_v)$  (usually infinite-dimensional) of these groups  $GL(n, k_v)$ . Lots of interesting number theory is hidden in *which* collections of representations are automorphic forms.

So the global Langlands conjectures assert that  $n$ -dimensional representations of the Galois group  $\Gamma$  of  $k$  give rise to irreducible representations of each  $GL(n, k_v)$ . Here is how this is supposed to work. Fix an algebraic closure  $\bar{k}_v$  of  $k_v$ , and let  $\Gamma_v = \text{Gal}(\bar{k}_v/k_v)$ . Then the algebraic closure of  $k$  in  $\bar{k}_v$  is an algebraic closure of  $k$ ; so the action of  $\Gamma_v$  on this closure defines an inclusion  $\Gamma_v \hookrightarrow \Gamma$ . Identifying this algebraic closure of  $k$  with the fixed one defining  $\Gamma$  requires a choice, so that this inclusion of  $\Gamma_v$  is defined only up to conjugation in  $\Gamma$ . Nevertheless, we get a well-defined restriction

$n$ -dimensional representations  $\tau$  of  $\Gamma \rightarrow n$ -dimensional representations  $\tau_v$  of  $\Gamma_v$ .

The correspondence from Galois representations to automorphic forms is supposed to arise from a correspondence

$n$ -dimensional representations  $\tau_v$  of  $\Gamma_v \rightarrow$  irreducible representations of  $GL(n, k_v)$ .

In this version, the automorphic forms have faded away, and we are left with a statement almost entirely about representation theory. The local Langlands conjecture is a refinement, replacing “ $n$ -dimensional representations of  $\Gamma_v$ ” by a somewhat larger set, and asking then for a *bijection* with the collection of irreducible representations of  $GL(n, k_v)$ . This local conjecture is a theorem, proved by Langlands around 1970 for archimedean places, and by Harris and Taylor around 2000 for non-archimedean places.

My plan is to introduce the representation theory of groups like  $GL(n, K_v)$ , especially  $GL(n, \mathbb{R})$ , from the perspective of the Langlands conjectures. There will be no required text, but there are certainly several sources that I hope to consult. These include Langlands paper “On the classification of representations of real algebraic groups”; Arthur’s paper “Unipotent automorphic representations: conjectures”; and my book with Adams and Barbasch *The Langlands Classification and Irreducible Characters for Real Reductive Groups*. In the end I hope to emphasize unitary representations of real groups.

**Time: MWF 1:00–2:00      Room: 2-102**

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