1. Representations of compact Lie groups

This is to write down some facts which are needed to do the homework. The main point is Proposition 1.4.

Setting is that we have a compact connected Lie group with a choice of maximal torus

$$G \supset T. \tag{1a}$$

Attached to the torus are two lattices

$$\begin{aligned} X^*(T) &= \mathrm{Hom}(T, U(1)) = \text{character lattice},\\ X_*(T) &= \mathrm{Hom}(U(1), T) = \text{cocharacter lattice}, \end{aligned} \tag{1b}$$

in duality by a pairing $\langle , \rangle$. Write

$$\mathfrak{g}_0 = \mathrm{Lie}(G), \quad \mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \tag{1c}$$

for the Lie algebra and its complexification; similar notation is used for other groups. The roots are defined by the weight decomposition of T in the adjoint representation

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in R} \mathfrak{g}(\alpha); \tag{1d}$$

then

$$R = R(G, T) \subset X^*(T)\backslash\{0\}, \qquad R^\vee = R^\vee(G, T) \subset X_*(T)\backslash\{0\}, \tag{1e}$$

where I have not recalled the definition of the coroots $R^\vee$.

It's often useful to have the rational vector spaces

$$X * (\mathbb{Q}) =_{\mathrm{def}} X^* \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad X_*(\mathbb{Q}) =_{\mathrm{def}} X_* \otimes_{\mathbb{Z}} \mathbb{Q}; \tag{1f}$$

these are dual vector spaces by the rational extension of $\langle , \rangle$. The following definitions can be made for X^* or for $X_*(\mathbb{Q})$. An element $\xi \in X_*(\mathbb{Q})$ is *regular* if and only if

$$\langle \alpha, \xi \rangle \neq 0 \qquad (\alpha \in R); \tag{1g}$$

otherwise ξ is *singular*. To each regular element ξ one can attach a *system of positive roots*

$$R^+(\xi) = \{\beta \in R \mid \langle \beta, \xi \rangle > 0\}. \tag{1h}$$

We fix henceforth a system of positive roots, called just R^+. Obviously R is the disjoint union of the positive and negative roots. A positive root is called *simple* if it is not the sum of two other positive roots. Write

$$\Pi = \{\alpha \in R^+ \mid \alpha \text{ is simple}\} \tag{1i}$$

for the set of simple roots.

Proposition 1.1. *Every positive root can be written uniquely as a sum with nonnegative integer coefficients of simple roots. The simple roots are a basis for the subspace of $X^*(\mathbb{Q})$ spanned by the roots, and a basis for the sublattice of X^* generated by the roots.*

A weight $\mu \in X^*(\mathbb{Q})$ is called *dominant* if

$$\langle \mu, \alpha^\vee \rangle \geq 0 \qquad (\alpha \in R^+); \tag{2}$$

it is enough to check this positivity for $\alpha \in \Pi$. In exactly the same way we can define *dominant coweights* in $X_*(\mathbb{Q})$.

Proposition 1.2. *Every orbit of the Weyl group on $X^*(\mathbb{Q})$ contains exactly one dominant weight.*

Definition 1.3. *Suppose μ and λ belong to X^*. We write*

$$\mu \preceq \lambda$$

if λ is equal to μ plus a nonnegative integer combination of positive roots.

Here is one description of the weights of an irreducible representation.

Proposition 1.4. *Suppose $\xi \in X^*$ is dominant, and $\pi(\xi)$ is the irreducible representation of G of highest weight ξ. Then the dominant weight $\mu \in X^*$ is a weight of $\pi(\xi)$ if and only if $\mu \preceq \xi$.*

If $\tau \in X^*$ is arbitrary, then τ is a weight of $\pi(\xi)$ if and only if the dominant conjugate $w\tau$ satisfies $w\tau \preceq \xi$. In this case it is also true that $\tau \preceq \xi$.

One has to be careful with quantifiers: if τ is not dominant, it can happen that $\tau \preceq \xi$, but nevertheless τ is not a weight of $\pi(\xi)$.

Here is another description of the weights.

Proposition 1.5. *Suppose $\xi \in X^*$ is dominant, and $\pi(\xi)$ is the irreducible representation of G of highest weight ξ. Then the set of weights of $\pi(\xi)$ is equal to the intersection of*

$$\textit{convex hull of } W \cdot \xi \subset X^*(\mathbb{Q})$$

with the lattice coset

$$\xi + \mathbb{Z}R$$

of translates of ξ by roots.