1. Describe completely the branching law from SU(2) to the binary icosahedral group $I^\sim$ of order 120. This means that you should say how to write the irreducible $m$-dimensional representation of $SU(2)$ as a combination of the nine irreducible representations $\pi_0 = \text{trivial}, \pi_1, \ldots, \pi_8 = \text{tautological}$. The labeling was explained in class. The McKay notes give a bijection between the nontrivial irreducibles and the simple roots of $E_8$; those simple roots are numbered 1–8 by Bourbaki.

Write $\sigma_m$ for the $m$-dimensional irreducible representation of $SU(2)$ (on the $m-1$st symmetric power of $\mathbb{C}^2$, although there is no need to know that description). In particular, the defining representation of $SU(2)$ is $\tau = \sigma_2$. (The letter $\tau$ stands for “tautological.”) In order to do this problem, you need to know two McKay graphs. The one for $SU(2)$ is

\[
\begin{array}{cccccccc}
\sigma_1 & \sigma_2 = \tau & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \ldots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots
\end{array}
\]

The graph for $I^\sim$ is

\[
\begin{array}{cccccccc}
\pi_0 & \pi_8 = \tau & \pi_7 & \pi_6 & \pi_5 & \pi_4 & \pi_3 & \pi_1 \\
1 & 2 & 3 & 4 & 5 & 6 & 4 & 2
\end{array}
\]

Recall that the numerical labels of the nodes are dimensions of irreducible representations, and that the meaning of each graph is

\[(1) \quad \alpha \otimes \tau = \sum_{\alpha - \beta} \beta.\]

In particular,

\[(2) \quad \sigma_m \otimes \tau = \sigma_{m-1} + \sigma_{m+1} \quad (m \geq 1).\]

(For $m = 1$, we need to interpret $\sigma_0$ as the zero representation in this formula.) For the binary icosahedral group, the graph represents nine different formulas not so easily compressed into one: for example,

\[\pi_4 \otimes \tau = \pi_5 + \pi_2 + \pi_3.\]
Formula (2) is an isomorphism of representations of $SU(2)$. If we restrict it to the binary icosahedral group, we get
\begin{equation}
(\sigma_m|_{I^-}) \otimes \tau = (\sigma_{m-1}|_{I^-}) + (\sigma_{m+1}|_{I^-}) \quad (m \geq 1).
\end{equation}
Equation (3) is an inductive procedure for computing the restriction of $\sigma_m$ to $I^-$. The base case $m = 1$ corresponds to the trivial representation of $SU(2)$, which restricts to the trivial representation of $I^-$: \[\sigma_1|_{I^-} = \pi_0.\]
In the inductive step, we assume $m \geq 2$, and that we already know the restrictions $\sigma_{m-1}|_{I^-}$ and $\sigma_m|_{I^-}$. Then we compute
\begin{equation}
(\sigma_{m+1}|_{I^-}) = (\sigma_m|_{I^-}) \otimes \tau - \sigma_m|_{I^-}.
\end{equation}
The first term on the right is computed from the McKay graph for the binary icosahedral group, together with our inductive knowledge of branching for $\sigma_m$; and the second we simply know by induction. The minus sign is at the level of multiplicities of irreducible representations (or, more formally, in the Grothendieck group of virtual representations of $I^-$). Here are the first few cases:
\begin{align*}
\sigma_2|_{I^-} &= ([\sigma_1|_{I^-}) \otimes \tau] - \sigma_0|_{I^-} \\
&= [\pi_0 \otimes \tau] - 0 = \pi_8 \\
\sigma_3|_{I^-} &= ([\sigma_2|_{I^-}) \otimes \tau] - \sigma_1|_{I^-} \\
&= [\pi_8 \otimes \tau] - \pi_0 \\
&= [\pi_0 + \pi_7] - \pi_0 = \pi_7
\end{align*}
Below I have tabulated the results of this calculation for $m \leq 30$. (I have written $\pi_j(d)$ when $\dim \pi_j = d$, in order to make it easier to see that the dimensions in row $m$ add to $m$.) Here are some periodicity properties which allow you to fill in the rest of the table.
(1) The multiplicity functions $m_{\sigma_m}(\pi_j)$ extend to all $m \in \mathbb{Z}$, by
\[m_{\sigma_{-m}}(\pi_j) = -m_{\sigma_m}(\pi_j).\]
The recursion formulas (3), etc., continue to hold.
(2) The multiplicities $m_{\sigma_m}(\pi_0), m_{\sigma_m}(\pi_7), m_{\sigma_m}(\pi_5), m_{\sigma_m}(\pi_2), m_{\sigma_m}(\pi_3)$ are all zero unless $m$ is odd.
(3) The multiplicities $m_{\sigma_m}(\pi_8), m_{\sigma_m}(\pi_6), m_{\sigma_m}(\pi_4), m_{\sigma_m}(\pi_1)$ are all zero unless $m$ is even.
(4) The multiplicity functions are “almost periodic” of period 60:
\[m_{\sigma_{m+60}}(\pi_j) = m_{\sigma_m}(\pi_j) + \dim \pi_j.\]
(5) The even multiplicity functions are “almost periodic” of period 30:
\[m_{\sigma_{2m+30}}(\pi_j) = m_{\sigma_{2m}}(\pi_j) + (\dim \pi_j)/2.\]
Branching law for $I^\sim$

<table>
<thead>
<tr>
<th></th>
<th>$\pi_0(1)$</th>
<th>$\pi_7(3)$</th>
<th>$\pi_5(5)$</th>
<th>$\pi_2(3)$</th>
<th>$\pi_3(4)$</th>
<th>$\pi_8(2)$</th>
<th>$\pi_6(4)$</th>
<th>$\pi_4(6)$</th>
<th>$\pi_1(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{-2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{-1}$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_3$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_5$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_8$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_9$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{10}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{11}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{12}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{13}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{14}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{15}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{16}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{17}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{18}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{19}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{20}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{21}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{22}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{23}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{24}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{25}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{26}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{27}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{28}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{29}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{30}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma_{31}$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>