1. 18.757 Homework 5

Due Tuesday, April 9.

These problems use the notes on classical groups on the class web site. In particular, the notes include a proof of what had been labeled "problem 1" when I wrote these on the board; so that one has been removed. I have added a new problem 4, which amounts to a hint for problem 5.

1. Suppose (π, V_π) is a finite-dimensional irreducible of $O(2p+1)$. Prove that the restriction of π to $SO(2p+1)$ is still irreducible. (This is true for representations over any field, and no continuity hypothesis is needed. But you can prove it only for continous complex representations if you like.)

2. The notes in (4.3) define a homomorphism from $U(n)$ to $SO(2n)$ using the standard identification $\mathbb{C} \simeq \mathbb{R}^2$. If

$$\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n, \tag{1a}$$

then we can get another identification

$$\mathbb{C}^n \xrightarrow{\phi_\epsilon} \mathbb{R}^{2n}, \qquad z_p = a_p + b_p i \mapsto \begin{cases} (a_p, b_p) & \epsilon_p = 0 \\ (a_p, -b_p) & \epsilon_p = 1. \end{cases} \tag{1b}$$

That is, we replace z_p by $\overline{z}_p$ whenever $\epsilon_p = 1$. The identification ϕ_ϵ defines a homomorphism

$$j_\epsilon \colon U(n) \to O(2n). \tag{1c}$$

Define

$$J_\epsilon = j_\epsilon(iI_n) \in O(2n). \tag{1d}$$

Suppose $0 \leqslant p \leqslant n$ is an integer, and $n = p + q$. Write

$$\epsilon(p,q) = (0, \ldots, 0, 1, \ldots, 1) \qquad (p \text{ zeros and } q \text{ ones}). \tag{1e}$$

a) Show that the image of j_ϵ is equal to the centralizer in $O(2n)$ of the element J_ϵ.
b) Show that the centralizer of $J_{\epsilon(n,0)} J_{\epsilon(p,q)}$ in $O(2n)$ is equal to $O(2p) \times O(2q)$.
c) Show that

$$j_{\epsilon(n,0)}(U(n)) \cap j_{\epsilon(p,q)}(U(n)) = [J_{\epsilon(p,0)} U(p)] \times [J_{\epsilon(0,q)} U(p)].$$

d) Now take $n = 2$. Show that

$$j_{\epsilon(2,0)}(SU(2)) \cap j_{\epsilon(1,1)}(SU(2)) = \{\pm I_4\} \subset SO(4).$$

e) Show that $j_{\epsilon(2,0)}(SU(2))$ and $j_{\epsilon(1,1)}(SU(2))$ commute with each other. (I don't know an easy way to see this. One possibility is to look at the Lie algebras.)
f) Conclude that

$$SO(4) \simeq [SU(2) \times SU(2)]/\{\pm I_2\},$$

the two-element subgroup embedded diagonally in the product.

3. Write $H_m(n)$ for the complex representation of $O(n)$ on complex-valued harmonic polynomials of degree m. Recall that $H_m(n)$ is irreducible or zero, of dimension $\binom{m+n-1}{n-1} - \binom{m+n-3}{n-1}$, and that

$$H_m(n)|_{O(n-1)} = \sum_{p=0}^{m} H_p(n-1)$$

for $n \geqslant 2$. In particular,

$$\dim H_m(O(4)) = (m+1)^2,$$
$$H_m(4)|_{O(3)} = H_m(3) \oplus H_{m-1}(3) \oplus \cdots \oplus H_0(3).$$

The formula for dimensions corresponding to the last statement is

$$(m+1)^2 = (2m+1) + (2m-1) + \cdots + 1.$$

Show that $H_m(4)$ remains irreducible on restriction to $SO(4)$.

4. The inclusion $j_{\epsilon(2,0)}$ of $SU(2)$ in $SO(4)$, and equations (4.4) in the notes, define an action of the unit quaternions on the sphere in $\mathbb{R}^4$; that is, on the unit quaternions. Show that this action is left multiplication. Show that the corresponding action defined using $j_{\epsilon(1,1)}$ is right multiplication by the inverse. (Recall that the inverse of a unit quaternion is equal to its conjugate.)

5. Show that $SU(2)$ has exactly one irreducible continuous complex representation V_m of dimension m for every integer $m \geqslant 1$.