1. 18.757 Homework 4

Due Thursday, March 14.

1.1. **Formal background.** Suppose that (π, V_π) is a irreducible representation of G over a field k, and that $D_\pi = \mathrm{Hom}_G(V_\pi, V_\pi)$ is the commuting algebra (endowed with the usual "reverse multiplication," so that V_π is a right vector space over D_π). If W is any representation of G over k, the multiplicity space and isotypic space are

$$W^\pi = \mathrm{Hom}_G(V_\pi, W), \qquad W(\pi) = \text{largest sum of copies of } V_\pi \text{ in } W.$$

Then W^π is a *left* vector space over D_π (by the right action on V_π), and there is an isomorphism

$$V_\pi \otimes_{D_\pi} W^\pi \to W(\pi), \qquad v \otimes A \mapsto A(v).$$

Using these (easy) facts, it's easy to make the following deductions.

(1) There is a one-to-one correspondence between G-invariant subspaces of $W(\pi)$ and (left) D_π-subspaces of W^π, $S \mapsto S^\pi$.

(2) There is a natural isomorphism

$$\mathrm{Hom}_G(W(\pi), U(\pi)) \simeq \mathrm{Hom}_{D_\pi}(W^\pi, U^\pi);$$

a map B on the left goes to the map on the right sending $A \in W^\pi$ to $B \circ A$.

In class I discussed the corresponding definitions for quotients of W:

$$W_\pi = \mathrm{Hom}_G(W, V_\pi),$$

$$W[\pi] = \varprojlim \left(\text{quotients isom to } \textit{direct products} \text{ of copies of } V_\pi\right).$$

Then W_π is a right vector space over D_π (by the right action on V_π). The natural map

$$W \to \mathrm{Hom}_{D_\pi}(W^\pi, V_\pi), \qquad w \mapsto (T \mapsto T(w))$$

(for $T \in W_\pi = \mathrm{Hom}_G(W, V_\pi)$) gives rise to an isomorphism

$$W[\pi] \to \mathrm{Hom}_{D_\pi}(W^\pi, V_\pi).$$

Notice that the right side is a direct product of copies of V_π, indexed by a D_π-basis of W^π.

1.2. **Homework problems for Thursday, March 14.** For these problems, Suppose V_π is a finite-dimensional irreducible representation of G, and D_π is as above. Recall that you proved in the second problem set that V_π^* is an irreducible representation of G; you may assume that the commuting algebra is again D_π, this time acting on the left on V_π^*.

1. Prove: $V_\pi \otimes_{D_\pi} V_\pi^*$ is an irreducible representation of $G \times G$.

2. Prove: $\mathrm{Hom}_{D_\pi}(V_\pi, V_\pi)$ is an irreducible representation of $G \times G$.

3. Calculate the commuting algebra of one of these two irreducible representations of $G \times G$.

4. Say as much as you can about the finite-dimensional irreducible k-representations of a product group $G \times H$, in terms of the irreducible representations of G and of H.