1. 18.757, Homework 1

Due Tuesday, February 12.

1. Write S^{n-1} for the unit sphere in $\mathbb{R}^n$, and $O(n)$ for the group of $n \times n$ real orthogonal matrices. Write

$$S^k(\mathbb{C})^n = \text{complex polynomial fns on } \mathbb{R}^n\text{, homog of degree } k.$$

Write

$$V = C(S^{n-1})$$

for the continuous complex-valued functions on the sphere, and V_{even} (respectively V_{odd} for the subspace of even (respectively odd) functions.

a) Show that restriction to the sphere defines inclusions

$$S^{2m}(\mathbb{C}^n) \hookrightarrow V_{\text{even}}, \quad S^{2m+1}(\mathbb{C}^n) \hookrightarrow V_{\text{odd}},$$

for all $m \geqslant 0$. Write W^k for the image of $S^k(\mathbb{C}^n)$; we discussed in class the fact that W^k is an $O(n)$-invariant subspace of V.

b) Show that $W^k \subset W^{k+2}$.

c) Show that the sum over k of W^k is a dense subspace of V; that is, that for any continuous function f on the sphere, and any $\epsilon > 0$, there is an $m \geqslant 0$ and a function

$$h \in W^{2m} + W^{2m+1}$$

such that $|h - f| < \epsilon$.

d) Write

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

for the Laplace operator. Obviously Δ defines a linear map

$$\Delta\colon S^k(\mathbb{C}^n) \to S^{k-2}(\mathbb{C}^n),$$

(almost obviously) respecting the action of $O(n)$. Define

$$H^k = \ker(\Delta|_{S^k(\mathbb{C}^n)}),$$

the space of *harmonic polynomials of degree* k. Similarly, we have a linear map

$$r^2\colon S^{k-2}(\mathbb{C}^n) \to S^k(\mathbb{C}^n),$$

also respecting the action of $O(n)$. Prove that

$$S^k \simeq \operatorname{im}(r^2) \oplus H^k.$$

(Hint: one approach is to recall that if $T\colon E \to F$ is a linear map between finite-dimensional inner product spaces, then

$$F = \operatorname{im}(T) \oplus \ker(T^*).)$$

It follows from the problem that

$$W^k \simeq H^k \oplus H^{k-2} \oplus H^{k-4} \cdots ,$$

an $O(n)$-invariant direct sum decomposition with $[k/2]$ terms, and that

$$\dim H^k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}.$$

2. (With thanks to Inna Entova-Aizenbud.)
   a) Prove (as claimed in class) that for $n = 2$, H^k (restricted to the unit circle) is spanned by $\cos(k\theta)$ and $\sin(k\theta)$.
   b) I said in class that H^k consists of even functions if k is even and odd functions if k is odd. But $\cos(k\theta)$ is *always* even, and $\sin(k\theta)$ is *always* odd. Explain.