18.745 Problem Set 8 **due in class Tuesday 5/5/15**

These problems refer to the notes on root systems (and vector root systems and root data) posted on the course web page; the main point will be for definitions and notation. In addition, you'll need two facts which I think were proved in class by Seth.

Proposition. *Suppose R is a root system with positive roots R^+ and simple roots Π. Then the Weyl group W is generated by the reflections s_α in simple roots $\alpha \in \Pi$. Every root is conjugate by W to a simple root.*

You also know from Seth's lectures that any root system is an orthogonal direct sum of simple root systems. The simple summands are the equivalence classes for the relation

$$\alpha \sim \beta \iff \langle \alpha, w\beta \rangle \neq 0 \qquad (\text{some } w \in W).$$

1. Because of the definition $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ for a root system, Lemma 5.3 of the notes says that if α and β are adjacent roots (meaning that $\langle \alpha, \beta \rangle \neq 0$) then the squared lengths of α and β differ by a factor of 1, 2, or 3. Suppose that R is a simple root system. Prove that *either*

1) all the roots have the same length, *or*
2) the roots have exactly two squared lengths, differing by a factor of 2 or 3.

In the case 1), the root system is called *simply laced*. In case 2), we can write

$$R = R_\ell \cup R_s, \quad \langle \alpha, \alpha \rangle = d\langle \beta, \beta \rangle \quad (\alpha \in R_\ell, \beta \in R_s).$$

The roots in R_ℓ are called *long*, and those in R_s are called *short*.

2. Suppose R is a simple root system with two root lengths; write

$$\Pi_\ell = \{\alpha_1, \ldots, \alpha_r\}, \quad \Pi_s = \{\beta_1, \ldots, \beta_s\}$$

for the long and short simple roots (with squared lengths differing by d). Prove that if

$$\alpha = \sum \lambda_i \alpha_i + \sum \sigma_j \beta_j$$

is any *long* root, then all the integers σ_j are divisible by d.

3. Suppose $R \subset E$ is a simple root system with one root length, and that δ is a graph automorphism of the Dynkin diagram of order $d > 1$. Assume that d is two or three (in fact that is automatic) and that *α is not adjacent to $\delta\alpha$ for any $\alpha \in R$.* Make δ into an automorphism of E (by applying the permutation δ to the basis Π of E). Let

$$D\colon E \to E^\delta, \qquad e \mapsto \frac{1}{d}\left(e + \delta \cdot e + \cdots + \delta^{d-1} \cdot e\right)$$

be the orthogonal projection. Prove that $(D(R), E^\delta)$ is a simple root system with two root lengths: the long roots R^δ, and the short roots $D(R - R^\delta)$.

4. Give an example of a simple root system $R \subset E$ with one root length, and a linear transformation δ of E of order two preserving R, such that $D(R)\backslash\{0\} \subset E^\delta$ is *not* a root system.

5. Prove that *every* simple root system with two root lengths arises by the construction in problem 3. (It's possible to solve this using the classification theorem, just writing things down case by case. Of course it's better to find an *a priori* argument not using the classification; the earlier problems are meant to be hints for doing that.