18.745 Problem Set 7 **due in class Tuesday 4/7/15**

(First problem corrected 3/29.) I may add to this list after class 3/31 or 4/2.

1. Suppose k is a field of *odd* characteristic p. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(2,k)$, with basis $\{X,H,Y\}$ as in the notes on representations of $\mathfrak{sl}(2)$. Write

$$\Omega = H^2 + 2H + 4YX = H^2 - 2H + 4XY$$

be the central element of the enveloping algebra defined in the notes. Suppose

$$\mu \in k, \quad \mu \notin \mathbb{F}_p$$

(that is, μ is in k but is *not* an integer); and

$$\kappa \in k, \quad \kappa \pm \mu \notin \mathbb{F}_p;$$

(that is, $\kappa \pm \mu$ is *not* an integer); and

$$A \in k^\times.$$

Prove that there is a unique irreducible representation (τ, V) of $\mathfrak{g}$ with the properties

$$\begin{aligned} &\tau(H) \text{ has eigenvalues } \{\mu + 2j \mid j \in \mathbb{F}_p\},\\ &\tau(\Omega) = (\kappa^2 + 2\kappa)I,\\ &\tau(X^p) \text{ has } A \text{ as an eigenvalue.}\end{aligned}$$

2. In the same setting as #1, find an element of the center of $U(\mathfrak{g})$ that is *not* a polynomial in Ω. (Hint: can you find an element that acts by scalars in the representation τ?)

3. Suppose k is a field of characteristic 2. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(2,k)$, with basis $\{X,H,Y\}$ as in the notes on representations of $\mathfrak{sl}(2)$.

a) Classify the irreducible 1-dimensional representations of $\mathfrak{g}$.

b) Suppose a, b, and c are in k, and $a \neq 0$. Prove that there is a two-dimensional irreducible representation (τ, V) of $\mathfrak{g}$ with the properties

$$\tau(H^2) = a^2 I, \quad \tau(X^2) = b^2 I, \quad \tau(Y^2) = c^2 I.$$

c) Suppose in addition that k is algebraically closed. Prove that every irreducible representation of $\mathfrak{g}$ is one of those you found in the first two parts.

Here's a metamathematical question for you to think about (but, in fairness to the grader, not to answer). Questions 1 and 3 mostly concerned p-dimensional representations of $\mathfrak{sl}(2)$ over a field of characteristic $p > 0$. Why are the cases $p = 2$ and p odd so different? Could I have reformulated Question 1 so that it applied also in characteristic two?