18.745 Problem Set 6 **due in class 3/17/15**

1. Let $\mathfrak{g}$ be a Lie algebra over a field k and V a representation of $\mathfrak{g}$. I proved in class that

$$H^0(\mathfrak{g}, V) \simeq V^{\mathfrak{g}} \tag{$H0$}$$

$$\begin{aligned} H^1(\mathfrak{g}, V) \simeq \{&\text{short exact sequences of Lie algebra representations} \\ &0 \to V \to E \to k \to 0\} \end{aligned} \tag{$H1$}$$

Prove that $H^2(\mathfrak{g}, V)$ may be identified with Lie algebras $\mathfrak{h}$ endowed with an abelian ideal V, and an isomorphism $\mathfrak{h}/V \simeq \mathfrak{g}$:

$$0 \to V \to \mathfrak{h} \to \mathfrak{g} \to 0. \tag{$H2$}$$

(Two such Lie algebras $\mathfrak{h}$ and $\mathfrak{h}'$ are identified if they are isomorphic by a map restricting to the identity on V and factoring to the identity on $\mathfrak{g}$.)

2. Prove that

$$H^1(\mathfrak{g}, k) \simeq \mathrm{Hom}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], k).$$

Explain how this identification is related to $(H1)$ in Problem 1.

3. Let $\mathfrak{g}$ be the three-dimensional complex Lie algebra of 3×3 skew-symmetric complex matrices, and let A be the associative algebra with 1 generated by $\mathfrak{g}$: the linear span of all products of skew-symmetric matrices, with the empty product called the identity matrix. Calculate the algebra A.

4. In the example presented in class 3/12 (like problem 3 with 2×2 matrices),

$$\dim A / \dim \mathfrak{g} = 2/1 = 2.$$

In problem 3, this ratio turns out to be bigger than 2. For each n, find an example of an n-dimensional complex vector space V and a nonzero Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, so that the associative algebra $A \subset \mathrm{End}(V)$ has

$$\dim A / \dim \mathfrak{g}$$

as large as possible. (The examples from class and problem 3 turn out to be largest possible for $n = 2$ and $n = 3$. If you can't prove your example is largest possible, you can still get partial credit depending on how large it is.)