18.745 Problem Set 4 **due in class 3/3/15**

The Jordan decomposition $T = T_s + T_n$ explained in Section 5.9 of the text works in exactly the same way over any algebraically closed field k. There is also a *multiplicative Jordan decomposition* $t = t_s t_u$ for invertible linear transformations t: the requirement is that t_s and t_u commute, that t_s be diagonal, and that t_u have all eigenvalues equal to one (*unipotent*). Notation is not being abused here: the additive and multiplicative Jordan decompositions have the same semisimple part when both are defined.

1. Suppose $(A, *)$ is a finite-dimensional algebra over an algebraically closed field k, and that $t \in \mathrm{Aut}(A)$. Prove that $t_s \in \mathrm{Aut}(A)$ (and therefore that $t_u \in \mathrm{Aut}(A)$ also).

2. Suppose $(A, *)$ is a finite-dimensional algebra over an algebraically closed field k, and that $T \in \mathrm{Der}(A)$. Prove that $T_s \in \mathrm{Der}(A)$ (and therefore that $T_n \in \mathrm{Der}(A)$ also).

3. Suppose that T is an $m \times m$ matrix with entries in a field k, and that $\overline{k}$ is an algebraic closure of k. By regarding T as a matrix over $\overline{k}$, we can define matrices T_s and T_n over $\overline{k}$. Are the entries of T_s and T_n in k? Say as much as you can about this question. (For example, you might prove that it's true unless k is a quadratic extension of $\mathbb{Q}$ with class number 1, and then give a counterexample in that case assuming the generalized Riemann hypothesis.)

4. Suppose $k \subset K$ are fields. If (V, B) is a k-vector space with a symmetric bilinear form B, explain how to get a symmetric bilinear form B_K on V_K. Prove that if $W \subset V$ is a subspace, then

$$(W_K)^\perp = (W^\perp)_K.$$

Prove that

$$\mathrm{Rad}(V, B)_K = \mathrm{Rad}(V_K, B_K).$$

The point of this exercise was fill in an omitted step in the proof of the Cartan Criterion (which is partly stated as Lemma 5.55 in the text:

Proposition. *Suppose V is a finite-dimensional vector space over a field k of characteristic zero, and $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a Lie subalgebra. Write B for the trace form*

$$B(X, Y) = \mathrm{tr}(XY)$$

on $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ is solvable if and only if

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathrm{Rad}(B|_{\mathfrak{g}}).$$

The exercise allows you to deduce this theorem for k from knowing it for an algebraic closure $\overline{k}$.