18.745 Problem Set 2 due 2/19/15

1. Classify the three-dimensional real Lie algebras $\mathfrak{g}$ containing an abelian ideal $\mathfrak{h}$ of dimension 2.

Hint: fix a basis $\{X_1, X_2\}$ of $\mathfrak{h}$, and let X_0 be any element of $\mathfrak{g}$ not belonging to H. Then $\{X_0, X_1, X_2\}$ is a basis of $\mathfrak{g}$, so $\mathfrak{g}$ will be determined by knowledge of the brackets of any pair of distinct basis vectors. Because $\mathfrak{h}$ is an ideal, bracketing with X_0 defines a linear transformation T of $\mathfrak{h}$. What are the possibilities for T?

2. Let V be a vector space over k endowed with a symplectic bilinear form ω. Recall that this means

$$\omega\colon V \times V \to k$$

is linear in each variable separately, and $\omega(v, v) = 0$ for any $v \in V$. Applying this to $v + w$ and using the bilinear requirement, we deduce that

$$\omega(v, w) = -\omega(w, v) \qquad (v, w \in V).$$

Recall that the *symplectic group* is defined to be

$$Sp(V) = \{g \in \mathrm{Aut}(V) \mid \omega(g \cdot v, g \cdot w) + \omega(v, w) = 0\};$$

of course the group product is composition of linear transformations. Recall also that the *symplectic Lie algebra* is defined to be

$$\mathfrak{sp}(V) = \{T \in \mathrm{End}(V) \mid \omega(Tv, w) + \omega(v, Tw) = 0\};$$

the Lie bracket is commutator of linear transformations.

In class I defined the *Weyl algebra* $A(V)$ to be the free associative algebra generated by V, modulo the relations

$$vw - wv = \omega(v, w) \qquad (v, w \in V).$$

The problem is to prove two things: first, that there is a natural group homomorphism

$$Sp(V) \to \mathrm{Aut}(A(V));$$

and second, that there is a natural Lie algebra homomorphism

$$\mathfrak{sp}(V) \to \mathrm{Der}(A(V)).$$

(The problem mentioned at the beginning of Thursday's class was the Clifford algebra version of the same question; but since I didn't get to defining the Clifford algebra in class, I'm replacing it with this one. In addition, I mentioned in class the additional problem of proving (when V is finite-dimensional and ω is nondegenerate) that these derivations of $A(V)$ are inner. But I am removing that question from this week's problem set.)

I will try to add to these notes over the weekend a little more explanation of the words "free associative algebra generated by V," but the formulation of the problems should be fixed.