18.745 Problem Set 1 due 2/10/15

(1) Suppose k is a field, and $k[t]$ is the algebra of polynomial functions in one variable. Calculate the group $\mathrm{Aut}(k[t])$ of automorphisms of this algebra.
(2) Calculate the vector space $\mathrm{Der}(k[t])$ of derivations of $k[t]$.

For the next two problems, A is a *finite-dimensional* algebra over the real numbers, with product $*$ (not assumed to be associative or anything like that). In particular, A is a finite-dimensional real vector space; so if $T \in \mathrm{Hom}(A, A)$ is any linear map, then $\exp(T) \in \mathrm{Hom}(A, A)$ is a well-defined invertible linear map.

(3) Suppose that T is a derivation of A. Prove that $\exp(tT)$ is an automorphism of A, for every $t \in \mathbb{R}$. (Hint: write down what you're supposed to prove; it says that two functions of t (with values in A) are the same. Find an ODE satisfied by each of these functions of t.)
(4) Suppose that $T \in \mathrm{Hom}(A, A)$ is any linear map. Assume that $\exp(tT)$ is an automorphism of A, for every $t \in \mathbb{R}$. Prove that T is a derivation of A.
(5) Suppose A is an algebra over any field k of characteristic zero, and that $T \in \mathrm{Der}(A)$. Assume that for some positive integer N, $T^N = 0$. Use T to construct some automorphisms of A.
(6) What does problem 5 tell you about problems 1 and 2?