18.745 Problem Set 1 due 2/10/15

(1) Suppose $k$ is a field, and $k[t]$ is the algebra of polynomial functions in one variable. Calculate the group $\text{Aut}(k[t])$ of automorphisms of this algebra.

(2) Calculate the vector space $\text{Der}(k[t])$ of derivations of $k[t]$.

For the next two problems, $A$ is a finite-dimensional algebra over the real numbers, with product $\ast$ (not assumed to be associative or anything like that). In particular, $A$ is a finite-dimensional real vector space; so if $T \in \text{Hom}(A, A)$ is any linear map, then $\exp(T) \in \text{Hom}(A, A)$ is a well-defined invertible linear map.

(3) Suppose that $T$ is a derivation of $A$. Prove that $\exp(tT)$ is an automorphism of $A$, for every $t \in \mathbb{R}$. (Hint: write down what you’re supposed to prove; it says that two functions of $t$ (with values in $A$) are the same. Find an ODE satisfied by each of these functions of $t$.)

(4) Suppose that $T \in \text{Hom}(A, A)$ is any linear map. Assume that $\exp(tT)$ is an automorphism of $A$, for every $t \in \mathbb{R}$. Prove that $T$ is a derivation of $A$.

(5) Suppose $A$ is an algebra over any field $k$ of characteristic zero, and that $T \in \text{Der}(A)$. Assume that for some positive integer $N$, $T^N = 0$. Use $T$ to construct some automorphisms of $A$.

(6) What does problem 5 tell you about problems 1 and 2?