18.700 final review problems

I’ll try to post some solutions Saturday 12/14.

1. Let \( A \) be the \( 4 \times 3 \) matrix of real numbers

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & -1 & 0 \\
1 & 0 & -1 \\
\end{pmatrix}.
\]

a) Find the reduced row-echelon form of \( A \).

b) Let \( C \) be the column vector \( \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} \). Find all solutions \( X \) of the equation \( AX = C \).

c) Calculate the rank of \( A \).____

d) Does \( A \) have a left inverse?____ If so, find one.

e) Does \( A \) have a right inverse?____ If so, find one.

2. a) Give an example of a \( 3 \times 2 \) real matrix \( A \) and a non-zero column vector \( C \in \mathbb{R}^3 \) with the property that the equation \( AX = C \) has infinitely many solutions.

b) What are the possible ranks for a matrix \( A \) satisfying the conditions in part (a)?____

3. In this problem the field is the complex numbers. Suppose \( A = \begin{pmatrix} 2 + i & 2 - i \\ 1 + i & 3 - i \end{pmatrix} \).

a) Find the characteristic equation of \( A \).

b) Find the eigenvalues of \( A \).

c) Find the eigenvectors of \( A \).

d) Find an invertible matrix \( P \) so that \( P^{-1}AP \) is diagonal.

4. Suppose that \( A \) is the \( 4 \times 4 \) real matrix

\[
A = \begin{pmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & 1 & -1 \\
1 & -1 & 0 & 1 \\
1 & 0 & -1 & -1 \\
\end{pmatrix}.
\]

a) Calculate \( \det A \).

b) Calculate the matrix \( A^t A \).

c) Calculate \( \det A^t A \).
5.  
   a) List all $2 \times 3$ matrices over the two-element field $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ which are in reduced row-echelon form and have rank 2. (There are exactly seven of them.)  
   b) How many $2 \times 3$ matrices are there over the five-element field $\mathbb{Z}/5\mathbb{Z}$ which are in reduced row-echelon form and have rank 2?________

6.  
   a) Find a $2 \times 5$ real row-echelon matrix $A$ with the property that the three vectors

   $$
   \begin{pmatrix}
   -4 \\
   0 \\
   -2 \\
   0 \\
   1
   
   \end{pmatrix},
   \begin{pmatrix}
   -5 \\
   0 \\
   -6 \\
   1 \\
   0
   
   \end{pmatrix},
   \begin{pmatrix}
   -7 \\
   1 \\
   0 \\
   0
   
   \end{pmatrix}
   $$

   are a basis for the null space of $A$.

   b) Find a $2 \times 5$ real row-echelon matrix $A$ with the property that the three vectors

   $$
   \begin{pmatrix}
   1 \\
   1 \\
   1 \\
   1
   
   \end{pmatrix},
   \begin{pmatrix}
   5 \\
   4 \\
   3 \\
   2
   
   \end{pmatrix},
   \begin{pmatrix}
   -1 \\
   1 \\
   1 \\
   1
   
   \end{pmatrix}
   $$

   are a basis for the null space of $A$.

7.  
   a) Apply the Gram-Schmidt process to the three row vectors

   $$
   v_1 = (1, 1, 1, 1, 1), v_2 = (5, 4, 3, 2, 1), v_3 = (-1, 1, -1, 1, -1)
   $$

   in order to obtain orthogonal vectors $w_1, w_2$ and $w_3$ with the same span.

   b) Beginning with the vectors $w_1, w_2, w_3$ of part (a), find an orthogonal basis

   $$
   w_1, w_2, w_3, w_4, w_5
   $$

   of $\mathbb{R}^5$. (Hint: apply the Gram-Schmidt process to the five vectors

   $$
   w_1, w_2, w_3, (1, 0, 0, 0, 0), (0, 1, 0, 0, 0).
   $$

   c) Find a $2 \times 5$ real matrix $B$ with the property that the three column vectors $w_1^t, w_2^t, w_3^t$ are a basis for the null space of $B$. (Hint: you can use as rows of $B$ some of the vectors you found in part (b).)

8. Suppose that $A$ is an $n \times n$ matrix over any field $k$, with the property that $A^3 = A$.
   a) Prove that any eigenvalue of $A$ must be 1, 0, or $-1$.
   b) Prove that $A$ has an eigenvalue.
   c) Give an example with $k = \mathbb{Z}/2\mathbb{Z}$ and $n = 2$ to show that $A$ does not have to be diagonalizable. (That is, find a two by two matrix $A$ with entries in $\mathbb{Z}/2\mathbb{Z}$ such that $A^3 = A$, but $A$ is not diagonalizable.)
   d) Suppose $k = \mathbb{R}$. Must $A$ be diagonalizable? (This is too hard, and really a bit outside the scope of the course. But it’s only for practice.)