18.700 final review problems

I'll try to post some solutions Saturday 12/14.

1. Let A be the 4×3 matrix of real numbers

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

a) Find the reduced row-echelon form of A.

b) Let C be the column vector $\begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$. Find all solutions X of the equation $AX = C$.

c) Calculate the rank of A.____

d) Does A have a left inverse?____ If so, find one.

e) Does A have a right inverse?____ If so, find one.

2.

a) Give an example of a 3×2 real matrix A and a non-zero column vector $C \in \mathbb{R}^3$ with the property that the equation $AX = C$ has infinitely many solutions.

b) What are the possible ranks for a matrix A satisfying the conditions in part (a)?____

3. In this problem the field is the complex numbers. Suppose $A = \begin{pmatrix} 2+i & 2-i \\ 1+i & 3-i \end{pmatrix}$.

a) Find the characteristic equation of A.

b) Find the eigenvalues of A.

c) Find the eigenvectors of A.

d) Find an invertible matrix P so that $P^{-1}AP$ is diagonal.

4. Suppose that A is the 4×4 real matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix}.$$

a) Calculate $\det A$.

b) Calculate the matrix $A^t A$.

c) Calculate $\det A^t A$.

5.

a) List all 2×3 matrices over the two-element field $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ which are in reduced row-echelon form and have rank 2. (There are exactly seven of them.)

b) How many 2×3 matrices are there over the five-element field $\mathbb{Z}/5\mathbb{Z}$ which are in reduced row-echelon form and have rank 2?________

6.

a) Find a 2×5 real row-echelon matrix A with the property that the three vectors

$$\begin{pmatrix} -4 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

are a basis for the null space of A.

b) Find a 2×5 real row-echelon matrix A with the property that the three vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

are a basis for the null space of A.

7.

a) Apply the Gram-Schmidt process to the three row vectors

$$v_1 = (1, 1, 1, 1, 1), v_2 = (5, 4, 3, 2, 1), v_3 = (-1, 1, -1, 1, -1)$$

in order to obtain orthogonal vectors w_1, w_2 and w_3 with the same span.

b) Beginning with the vectors $w_1, w_2,$ and w_3 of part (a), find an orthogonal basis

$$w_1, w_2, w_3, w_4, w_5$$

of $\mathbb{R}^5$. (Hint: apply the Gram-Schmidt process to the five vectors

$$w_1, w_2, w_3, (1, 0, 0, 0, 0), (0, 1, 0, 0, 0).)$$

c) Find a 2×5 real matrix B with the property that the three column vectors w_1^t, w_2^t, and w_3^t are a basis for the null space of B. (Hint: you can use as rows of B some of the vectors you found in part (b).)

8. Suppose that A is an $n \times n$ matrix over any field k. with the property that $A^3 = A$.

a) Prove that any eigenvalue of A must be 1, 0, or -1.

b) Prove that A has an eigenvalue.

c) Give an example with $k = \mathbb{Z}/2\mathbb{Z}$ and $n = 2$ to show that A does not have to be diagonalizable. (That is, find a two by two matrix A with entries in $\mathbb{Z}/2\mathbb{Z}$ such that $A^3 = A$, but A is not diagonalizable.)

d) Suppose $k = \mathbb{R}$. Must A be diagonalizable? (This is too hard, and really a bit outside the scope of the course. But it's only for practice.)