18.700 Problem Set 9

Due in class Tuesday December 2; late work will not be accepted. Your work on graded problem sets should be written entirely on your own, although you may consult others before writing.

1. (8 points) Suppose \( V \) is a real or complex inner product space. A linear map \( S \in \mathcal{L}(V) \) is called skew-adjoint if \( S^* = -S \). Suppose \( V \) is complex and finite-dimensional, and \( S \) is skew-adjoint. Show that the eigenvalues of \( S \) are all purely imaginary (that is, real multiples of \( i \)) and that there is an orthogonal direct sum decomposition

\[
V = \bigoplus_{\lambda \in \mathbb{R}} V_{i\lambda}.
\]

2. (16 points) Suppose \( V \) is an \( n \)-dimensional real vector space, and \( S \) is a skew-adjoint linear transformation of \( V \).

a) Show that \( Sv \) is orthogonal to \( v \) for every \( v \in V \).

b) Show that every eigenvalue of \( S^2 \) is a real number less than or equal to zero.

c) Show (still assuming that \( S \) is skew-adjoint) that there is an orthonormal basis of \( V \) in which the diagonal blocks of the matrix of \( S \) are all either (0) or

\[
\begin{pmatrix}
0 & -\lambda \\
\lambda & 0
\end{pmatrix}
\]

with \( \lambda > 0 \).

d) Suppose (still assuming \( S \) is skew-adjoint) that \( S^2 = -I \) (the negative of the identity operator on \( V \)). Show that we can make \( V \) into a complex inner product space, by defining scalar multiplication as

\[
(a + bi)v = av + Sv
\]

and the complex inner product as

\[
\langle v, w \rangle_C = \langle v, w \rangle - i \langle Sv, w \rangle.
\]

What is the dimension of \( V \) as a complex vector space?