1. (8 points) Suppose that we are given three polynomials

\[ p_2(x) = ax^2 + bx + c, \quad p_1(x) = dx + e, \quad p_0(x) = f \]

with real coefficients. This problem is about the differential operator

\[ D = p_2(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_0(x). \]

a) Explain why the operator \( D \) preserves the \( m + 1 \)-dimensional space \( P_m(\mathbb{R}) \) of real polynomials of degree less than or equal to \( m \).

The operation \( \frac{d^2}{dx^2} \) lowers the degree of a polynomial by at least two; then multiplication by the quadratic \( p_2 \) raises it by at most two. So altogether the first term of \( D \) preserves \( P_m \). Exactly the same argument applies to the other two terms.

b) Find diagonal entries of the matrix of \( D \) in the basis \( (1, x, x^2, \ldots, x^m) \).

(Your answers will depend on the constants \( a, b, c, d, e, f \).)

We have

\[
D(x^k) = k(k - 1)p_2(x)x^{k-2} + kp_1(x)x^{k-1} + p_0(x)x^k \\
= k(k - 1)(ax^2 + bx + c)x^{k-2} + k(dx + e)x^{k-1} + f x^k \\
= [ak(k - 1) + dk + f]x^k + [bk(k - 1) + ek]x^{k-1} + cx^{k-2}.
\]

Therefore column \( k - 1 \) of the matrix (corresponding to the basis vector \( x^k \)) has three nonzero entries: \( [ak(k - 1) + dk + f] \) in row \( k - 1 \), \( [bk(k - 1) + ek] \) in row \( k - 2 \), and \( ck(k - 1) \) in row \( k - 3 \). In particular, the diagonal entries are

\[
ak(k - 1) + dk + f = ak^2 + (d - a)k + f \quad (0 \leq k \leq m).
\]

c) Suppose that the constants satisfy \( 0 < a < d \). Prove that \( D \) is diagonalizable on \( P_m(\mathbb{R}) \).

Part (b) showed that the matrix of \( D \) is upper triangular, so its eigenvalues are precisely its diagonal entries:

\[
ak^2 + (d - a)k + f, \quad 0 \leq k \leq m.
\]

Since \( a \) and \( d - a \) are strictly positive, both \( ak^2 \) and \( (d - a)k \) are strictly increasing for \( k \geq 0 \). Therefore these \( m + 1 \) eigenvalues are all distinct. By Proposition 5.20 in the text, \( D \) is diagonalizable.

d) Consider the special case

\[ D = (x^2 - x) \frac{d^2}{dx^2} + (2x - 1) \frac{d}{dx}. \]

Find a basis of eigenvectors of \( D \) on the four-dimensional space \( P_3(\mathbb{R}) \).
In this case $a = 1$ and $d = 2$, so the eigenvalues are $k^2 + k$, or 0, 2, 6, and 12. The matrix of $D$ is
\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 2 & -4 & 0 \\
0 & 0 & 6 & -9 \\
0 & 0 & 0 & 12
\end{pmatrix}
\]
The hardest eigenvalue problem is the last one:
\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 2 & -4 & 0 \\
0 & 0 & 6 & -9 \\
0 & 0 & 0 & 12
\end{pmatrix}
\begin{pmatrix}
 r \\
 s \\
t \\
u
\end{pmatrix}
= 12
\begin{pmatrix}
 r \\
 s \\
t \\
u
\end{pmatrix},
\]
or as a system of equations
\[
\begin{align*}
-12r - s &= 0 \\
-10s - 4t &= 0 \\
-6t - 9u &= 0 \\
0 &= 0.
\end{align*}
\]
This system is in row-echelon form; the free variable is $u$, which can therefore be anything, and the others are
\[
t = -3u/2, \quad s = 3u/5, \quad r = -u/20.
\]
That is, a basis for the eigenspace (taking $u = 1$) is
\[
x^3 - (3/2)x^2 + (3/5)x - 1/20 \quad (\lambda = 12).
\]
Similarly, bases for the other eigenspaces are
\[
x^2 - x + 1/6 \quad (\lambda = 6), \quad x - 1/2 \quad (\lambda = 2), \quad 1 \quad (\lambda = 0).
\]

2. (8 points) This problem is about the inner product space $C[0, 1]$ of real-valued continuous functions on the interval $[0, 1]$, with inner product
\[
\langle p, q \rangle = \int_0^1 p(x)q(x)dx.
\]
It’s like the example in the text about finding a good polynomial approximation to $\sin(x)$.
a) Let $U = P_3(\mathbb{R})$ be the four-dimensional subspace of $C[0, 1]$ consisting of polynomials of degree less than or equal to three. Apply the Gram-Schmidt process described in the notes on orthogonal bases on the course web site to convert the basis $(1, x, x^2, x^3)$ of $U$ to an orthogonal basis having the same span. (I suggest the notes because it’s easier than following the text.)

The procedure is
\[
f_1 = v_1 = 1;
\]
\[ f_2 = v_2 - \frac{\langle v_2, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 = x - \frac{1/2}{1} \cdot 1 = x - 1/2. \]

Here and in many places we use

\[ \langle x^p, x^q \rangle = \int_0^1 x^{p+q} \, dx = \left[ \frac{x^{p+q+1}}{p+q+1} \right]_0^1 = \frac{1}{p+q+1}. \]  

(\*)

Next, we have

\[ f_3 = v_3 - \frac{\langle v_3, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 - \frac{\langle v_3, f_2 \rangle}{\langle f_2, f_2 \rangle} f_2 = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 - \frac{\langle x^2, x - 1/2 \rangle}{\langle x - 1/2, x - 1/2 \rangle} \cdot (x - 1/2) = x^2 - 1/3 \cdot 1 - 1/4 - 1/6 \cdot (x - 1/2) = x^2 - 1/3 - x + 1/2 = x^2 - x + 1/6. \]

Finally,

\[ f_4 = v_4 - \frac{\langle v_4, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 - \frac{\langle v_4, f_2 \rangle}{\langle f_2, f_2 \rangle} f_2 - \frac{\langle v_4, f_3 \rangle}{\langle f_3, f_3 \rangle} f_3 = x^3 - \frac{\langle x^3, 1 \rangle}{\langle 1, 1 \rangle} \cdot f_1 - \frac{\langle x^3, x - 1/2 \rangle}{\langle x - 1/2, x - 1/2 \rangle} \cdot f_2 = x^3 - \frac{\langle x^3, x^2 - x + 1/6 \rangle}{\langle x^3, x^2 - x + 1/6, x^3, x^2 - x + 1/6 \rangle} \cdot f_3 = x^3 - 1/4 \cdot 1 - 1/5 - 1/8 \cdot (x - 1/2) - \frac{(1/6 - 1/5 + 1/24)}{1/180} \cdot (x^2 - x + 1/6) = x^3 - 1/4 - (9/10)(x - 1/2) - (3/2)(x^2 - x + 1/6) \]

\[ f_4 = x^3 - (3/2)x^2 + (3/5)x - (1/20). \]

b) Define \( f(x) = \sqrt{x} \), regarded as a function in \( C[0, 1] \). Calculate the orthogonal projection \( p_U(f) \) of \( f \) on the subspace \( U \). (Your answer should be a cubic polynomial in \( x \) with rational numbers as coefficients.)

According to the formula 6.35' from the notes,

\[ P_U(f) = \frac{\langle f, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 + \frac{\langle f, f_2 \rangle}{\langle f_2, f_2 \rangle} f_2 + \frac{\langle f, f_3 \rangle}{\langle f_3, f_3 \rangle} f_3 + \frac{\langle f, f_4 \rangle}{\langle f_4, f_4 \rangle} f_4. \]

(\**) 

The inner products on top are pretty easy to calculate using the formula (\*) from the previous part, which also works for nonnegative half integers \( p \) and \( q \). We calculated all the inner products on the bottom in the previous part, except for \( f_4 \); and that one is just a little painful:

\[ \langle f_1, f_1 \rangle = 1, \quad \langle f_2, f_2 \rangle = 1/12, \quad \langle f_3, f_3 \rangle = 1/180, \quad \langle f_4, f_4 \rangle = 1/2800. \]
Therefore (**) becomes

\[ P_U(f) = \frac{2/3}{1} f_1 + \frac{2/5 - 1/3}{1/12} f_2 + \frac{2/7 - 2/5 + 1/9}{1/180} f_3 + \frac{2/9 - 3/7 + 6/25 - 1/30}{1/2800} f_4 \]

\[ = (2/3) \cdot 1 + (4/5) \cdot (x - 1/2) - (4/7)(x^2 - x + 1/6) + (8/9)(x^3 - 3/2x^2 + 3/5x - 1/20). \]

c) Make a little table displaying the values of \( f(x) \) and \( P_U(f)(x) \) for at least the values \( x = 0, 1/4, 1/2, 3/4, 1 \), to three decimal places.

I’ll start with a matrix whose \( m \)th column gives the values of \( f_m(x) \) at these five values of \( x \):

\[
V = \begin{pmatrix}
1 & -1/2 & 1/6 & -1/20 \\
1 & -1/4 & -1/48 & 7/320 \\
1 & 0 & -1/12 & 0 \\
1 & 1/4 & -1/48 & -7/320 \\
1 & 1/2 & 1/6 & 1/20
\end{pmatrix}.
\]

To get the values of \( P_U(f) \) at these five values of \( x \), we simply apply the matrix \( V \) of values to the vector \( c = \begin{pmatrix} 2/3 \\ 4/5 \\ -4/7 \\ 8/9 \end{pmatrix} \) of coefficients (in the expansion of \( f \) in terms of \( f_m \)). The result is

\[
Vc = \begin{pmatrix}
1 & -1/2 & 1/6 & -1/20 \\
1 & -1/4 & -1/48 & 7/320 \\
1 & 0 & -1/12 & 0 \\
1 & 1/4 & -1/48 & -7/320 \\
1 & 1/2 & 1/6 & 1/20
\end{pmatrix} \begin{pmatrix} 2/3 \\ 4/5 \\ -4/7 \\ 8/9 \end{pmatrix} = \begin{pmatrix}
8/63 \\
251/504 \\
421/504 \\
64/63
\end{pmatrix}.
\]

(I did the arithmetic by hand; if you got different answers, especially from a computer program, I’d be happy to hear about it.) So here is the table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( P_U(f)(x) )</th>
<th>decimal value</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.000</td>
<td>8/63</td>
<td>0.127</td>
<td>.127</td>
</tr>
<tr>
<td>0.25</td>
<td>0.500</td>
<td>251/504</td>
<td>0.498</td>
<td>-.002</td>
</tr>
<tr>
<td>0.50</td>
<td>0.707</td>
<td>5/7</td>
<td>0.714</td>
<td>.007</td>
</tr>
<tr>
<td>0.75</td>
<td>0.866</td>
<td>421/504</td>
<td>0.835</td>
<td>-.031</td>
</tr>
<tr>
<td>1.00</td>
<td>1.000</td>
<td>64/63</td>
<td>1.016</td>
<td>.016</td>
</tr>
</tbody>
</table>

d) Find some relationship between your answers to 1(d) and 2(a). Make a guess about how to generalize that relationship to higher degree polynomials.
The polynomials in 1(d) and 2(a) are exactly the same. (Even if you chose different eigenvectors in 2(a), you would have to get polynomials proportional to those in 2(a).) The natural guess is that if we apply the Gram-Schmidt process to the basis \((1, x, \ldots, x^m)\), we get the eigenvectors for \(D\) for the eigenvalues \(0^2 + 0, 1^2 + 1, \ldots, m^2 + m\). This guess turns out to be correct. I’ll say a bit more about why that’s true in class. The key fact is that if \(p\) and \(q\) have lots of derivatives on \([0, 1]\), then
\[
\langle Dp, q \rangle = \langle p, Dq \rangle. \tag{SA}
\]
(The label \(SA\) stands for “Self-Adjoint.”) The reason this is true is integration by parts, which guarantees that
\[
\langle \frac{dp}{dx}, q \rangle = -\langle p, \frac{dq}{dx} \rangle + p(1)q(1) - p(0)q(0). \tag{IP}
\]
(The label is “Integration by Parts.”) There’s also a very easy formula
\[
\langle xp, q \rangle = \langle p, xq \rangle. \tag{IM}
\]
( Label means “Integral Multiplication.”) If you apply \((IP)\) and \((IM)\) repeatedly to the definition of \(\langle Dp, q \rangle\), eventually all the “boundary terms” like \(p(1)q(1)\) cancel, and you arrive at \(\langle p, Dq \rangle\).

Using \((SA)\), it’s easy to deduce that
eigenvectors of \(D\) for distinct eigenvalues are orthogonal. \(\tag{EO}\)
(This one means “Eigenvector orthogonality.”) The idea of 1(d) was to start with \(x^m\), then add lower degree terms until you get an eigenvector for \(D\), of eigenvalue \(m^2 + m\). The idea of 2(a) was to start with \(x^m\), then add lower degree terms until you get something orthogonal to all the lower powers of \(x\). Can you see why \((EO)\) says that these two processes have to lead to the same answer?

The polynomials in 1(d) are (almost) called Legendre polynomials, except that you should replace \(x\) by \((y + 1)/2\) and multiply by some scalar to get a Legendre polynomial. (The reason is that Legendre was clever enough to work with functions on the interval \([-1, 1]\) instead of \([0, 1]\), so all his formulas look better.) There are many amazing facts about Legendre polynomials, including Rodrigues’ formula
\[
f_{n+1}(x) = \frac{n!}{(2n)!} \frac{d^n}{dx^n} (x^2 - x)^n.
\]
You should check that this is correct for \(n = 0, 1, 2,\) and 3.