18.700 Problem Set 3

Due in class *Monday October 7*; late work will not be accepted. Your work on graded problem sets should written entirely on your own, although you may consult others before writing.

1. (3 points) Give an example of a 3×3 matrix A of real numbers whose reduced row-echelon form is
$$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$$
and such that every entry of A is a nonzero integer.

2. (3 points) The finite field $\mathbb{F}_9$ contains $\mathbb{Z}/3\mathbb{Z}$ and an element I'll call x satisfying $x^2 + 1 = 0$. Using this fact, write down the 9×9 multiplication table for $\mathbb{F}_9$.

3. (6 points) Suppose p is any prime number. Imitating the complex numbers, you can define a set of p^2 elements with addition and multiplication:
$$R_p = \{a + bi \mid a, b \in \mathbb{Z}/p\mathbb{Z}\}$$
$$(a + bi) + (c + di) =_{\text{def}} (a + c) + (b + d)i,$$
$$(a + bi)(c + di) =_{\text{def}} (ac - bd) + (ad + bc)i$$
The associative, commutative, and distributive laws are all inherited by R_p from the Gaussian integers $m + ni$ (with m and n in $\mathbb{Z}$) and so are the additive and multiplicative identities and additive inverses. *So R_p is a field if and only if it has multiplicative inverses.*

a) For the prime numbers $p = 2$, 3, 5, explain why R_p is or is not a field.

b) Prove that R_{53} is *not* a field. (Hint: $53 = 7^2 + 2^2$.)

c) Explain your best guess about whether R_{251} is a field. (For example, you might say, "we found that R_p was not a field for the odd primes 3, 5, and 53, so probably R_p is not a field for any odd prime p.")

4. (3 points) (Based on Axler, page 69, exercise 22). Suppose U is a finite-dimensional vector spaces, that $S \in \mathcal{L}(V, W)$, and that $T \in \mathcal{L}(U, V)$. Prove that
$$\dim \operatorname{null}(ST) = \dim \operatorname{null}(T) + \dim (\operatorname{range}(T) \cap \operatorname{null}(S)).$$

5. (3 points) Give an example of problem 4 with $U = V = W = \mathbb{R}^2$, with null(S) and null(T) both one-dimensional, but null(ST) *not* 2-dimensional.

6. (3 points) Suppose $T \in \mathcal{L}(V, W)$, and that V is finite-dimensional. (You can if you wish use things stated in class about left and right inverses; that isn't necessary, but it could help.)

a) Show that null$(T) = \{0\}$ if and only if for each linearly independent list $(v_1, \ldots, v_p)$ in V, $(Tv_1, \ldots, Tv_p)$ is linearly independent in W.

b) Show that range$(T) = W$ if and only if for each spanning list $(v_1, v_2, \ldots, v_q)$ in V, $(Tv_1, \ldots, Tv_q)$ is a spanning list in W.

c) Prove T is invertible if and only if T takes each basis of V to a basis of W.