1. (3 points) Consider the set of complex numbers 
\[ G = \{a + bi \mid a, b \in \mathbb{Q}\}. \]
(The \( G \) stands for Gauss; these numbers might be called Gaussian rational numbers, although I don’t know if they actually are.) Is \( G \) a field (with the same addition and multiplication operations as in \( \mathbb{C} \))? For a question like this, you should either explain why all the axioms for a field are satisfied (you can assume that they hold for \( \mathbb{C} \)), or else explain why one of the axioms fails. A few sentences could be enough to write.

The answer is yes. The rules for adding and multiplying complex numbers \( a + bi \) involve just addition, subtraction, and multiplication of the coefficients \( a \) and \( b \). Since \( \mathbb{Q} \) is closed under addition and multiplication, so is \( G \). The identities 0 and 1 for \( \mathbb{C} \) belong to \( G \). The formulas for inverses
\[
-(a + bi) = -a - bi, \quad (a + bi)^{-1} = a/(a^2 + b^2) - bi/(a^2 + b^2)
\]
give rational coefficients for rational \( a \) and \( b \); so \( G \) has inverses. The remaining axioms for a field (commutativity, associativity, and distributive law) are inherited from \( \mathbb{C} \).

2. (3 points) Consider the set of complex numbers
\[ M = \{re^{2\pi i \theta} \mid r \in \mathbb{Q}, \ \theta \in (1/4)\mathbb{Z}\}. \]

Is \( M \) a field?

The answer is no. Because \( e^{2\pi i/4} = e^{i\pi/2} = i \), the elements of \( M \) are “rational number times some power of \( i \).” That is,
\[ M = \{r \mid r \in \mathbb{Q}\} \cup \{si \mid s \in \mathbb{Q}\}. \]

It’s easy to check that this set is closed under multiplication: for example, if \( r \) and \( s \) are rational numbers, then
\[ (ri)(si) = -rs \in \mathbb{Q}. \]

It also includes 0 and 1 and has both multiplicative and additive inverses. But \( M \) is not closed under addition: 1 and \( i \) are both in \( M \), but 1 + \( i \) is not. So \( M \) is not a field.

Remark: this problem was a mistake. I intended to allow linear combinations like \( re^{i\pi/2} + se^{i\pi} \), so that \( M \) would have been exactly the same as the set \( G \) in the first problem, and it would have been a field. Oops.

3. (3 points) Consider the set of complex numbers
\[ M = \{re^{2\pi i \theta} \mid r, \ \theta \in \mathbb{Q}\}. \]

Is \( M \) a field?
Again the answer is no. It’s easy to check that \( M \) is closed under multiplication, because
\[
re^{i\theta} \cdot r'e^{i\theta'} = rr' e^{i(\theta + \theta')},
\]
and if \( r \) and \( r' \) and \( \theta \) and \( \theta' \) are all rational, so are \( rr' \) and \( \theta + \theta' \). Furthermore \( M \) includes 0 and 1 and has both multiplicative and additive inverses. But \( M \) is (for a slightly different reason than in Problem 2) not closed under addition. To prove that, it isn’t enough to say “it doesn’t appear to be closed under addition;” you need to write down two elements of \( M \) whose sum does not belong to \( M \). There are lots of ways to do that; here is one. Take
\[
m_1 = e^{2\pi i/8} = \cos(\pi/4) + i \sin(\pi/4) = (\sqrt{2}/2)(1 + i),
\]
\[
m_2 = e^{-2\pi i/8} = \cos(-\pi/4) + i \sin(-\pi/4) = (\sqrt{2}/2)(1 - i).
\]
Then \( m_1 + m_2 = \sqrt{2} \), a complex number of length \( \sqrt{2} \). I claim that it does not belong to \( M \). The reason is that \( re^{2\pi i\theta} \) has length \( |r| \), so the only way it could be equal to \( m_1 + m_2 \) is if \( r = \pm \sqrt{2} \). But \( \sqrt{2} \) is irrational; so \( m_1 + m_2 \) is not in \( M \).

4. (3 points) The vector space \( V = (\mathbb{F}_2)^2 \) has exactly four vectors \( (0,0), (0,1), (1,0), \) and \( (1,1) \); so \( V \) has exactly \( 2^4 = 16 \) subsets. How many of these 16 subsets are linearly independent? How many bases does \( V \) have? For a question like this, you might write some words explaining why some kinds of subset cannot possibly be linearly independent (“the vector \((1,1)\) is in the pay of Big Oil, and so cannot be part of any linearly independent set”). After this you might be left with just a few cases; you could perhaps say a few words about why each of these is or is not linearly independent.

No set of vectors including \((0,0)\) can be linearly independent (text, middle of page 24). So the only candidates are the 8 subsets of \( \{ (0,1), (1,0), (1,1) \} \). The empty set is linearly independent, and so is any of the three one-element subsets (text, bottom of page 24). The only scalar multiples of \( v \) (in case \( F = \mathbb{F}_2 \)) are \( v \) and \( 0 \); so according to the text on page 24, any one of the three two-element subsets is linearly independent. That leaves only the three-element subset, which is linearly dependent because of the relation
\[
(0,1) + (1,0) + (1,1) = (0,0).
\]
The conclusion is that the linearly independent subsets are the 7 proper subsets of the 3 nonzero vectors.

A basis of \( V \) must have two elements, and any two-element linearly independent set is a basis (text, Theorem 2.14 and Proposition 2.17); so the three bases are the three two-element sets of nonzero vectors.