18.440 Problem Set 9

Due in class Wednesday November 19; late work will not be accepted. You can discuss problems with anyone, but you should write solutions entirely on your own.

1. (20 points) Suppose X is a random variable uniformly distributed over $[-1,1]$. Calculate the *kth moment*

$$m_k(X) = E(X^k),$$

for any integer $k \geq 0$. (Hint: even and odd values of k are different.)

2. (20 points) Suppose X and Y are independent random variables, and that all the higher moments $m_p(X)$ and $m_q(Y)$ are defined. Find a formula for the kth moment $m_k(X+Y)$ in terms of the various $m_p(X)$ and $m_q(Y)$. (Hint: the first example is the formula $E(X+Y) = E(X) + E(Y)$ for the first moments.)

3. (30 points) Suppose X_1, X_2, X_3, ... are independent random variables, each uniformly distributed over the interval $[-1,1]$. Define

$$S_n = X_1 + X_2 + \cdots + X_n,$$

a random variable taking values in $[-n,n]$.

a) Calculate $E(S_n)$.

b) Calculate $Var(S_n)$.

c) Calculate the kth moment $m_k(S_2)$. (Notice that this only asks about $S_2 = X_1 + X_2$. You should use the formulas you found in problems 1 and 2. Even and odd k are different. You can use the fact that

$$\sum_{p=1}^{q} \binom{2q}{2p-1} = 2^{2q-1}.)$$

4. (10 points) For the random variables in Problem 3, calculate probability density functions f_{S_k} for $k = 1$, 2, and 3. Graph them carefully enough to show that f_{S_2} has a derivative everywhere.

5. (20 points) Suppose f_X is the probability density function for a random variable uniformly distributed over $[-1, 1]$, and f is any continuous function. The point of this problem is to show that the convolution

$$(f * f_X)(x) = \int_{-\infty}^{\infty} f(t) f_X(x - t)\, dt$$

has a derivative.

a) Show that

$$(f * f_X)(x) = \frac{1}{2} \int_{x-1}^{x+1} f(t)\, dt.$$

b) Show that

$$(f * f_X)(x + \Delta x) - (f * f_X)(x) = \frac{1}{2} \int_{x+1}^{x+1+\Delta x} f(t)\, dt - \frac{1}{2} \int_{x-1}^{x-1+\Delta x} f(t)\, dt.$$

(This formula is easy to interpret as long as $\Delta x > 0$. You can just look at that case if you prefer. For negative values of Δx, you need to use conventions from calculus like $\int_a^b = -\int_b^a$; then the formula is still true.)

c) Show that

$$\frac{d}{dx}[(f * f_X)](x) = \frac{1}{2}(f(x + 1) - f(x - 1)).$$

d) Suppose g is a function on $\mathbb{R}$ having a continuous derivative and vanishing outside some finite interval. Explain why it ought to be true that

$$\frac{d}{dx}[f * g] = f * \frac{dg}{dx}.$$

(This formula really is exactly true, but I'm only asking for a plausibility argument.) Can you connect this formula to the one in part (c)?