18.440 Problem Set 8

Due in class Wednesday November 12; late work will not be accepted. You can discuss problems with anyone, but you should write solutions entirely on your own.

1. (40 points) Suppose you shoot at a disc of radius one, with shots uniformly distributed over the disc. Think of the sample space as

$$S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\},$$

and the probability of $E \subset S$ as (area of E)/π. Let X and Y be the random variables giving the coordinates of the points you hit.

a) Calculate the joint cumulative distribution function

$$F_{XY}(a, b) = P(X \leq a, Y \leq b).$$

b) Calculate the joint probability density function $f_{XY}(a, b)$.

c) Calculate $E(|X|)$, the expected value of the absolute value of the x coordinate.

d) Explain why the answer to (c) should be less than $R/2$.

2. (40 points) The town of Podunk, Massachusetts serves as the postal address for the Academy for Delinquent Preteens (a boys' school) and the Worcester Inspirational Learning Generation (a girls' school). Each day the Podunk Bus Emporium sends its single van to one of the two schools, choosing its destination at random: the Academy is chosen with probability p, and the Learning Generation with probability $1-p$. On arriving at the chosen school, the van picks up whatever students wish to visit the post office. The number of such students is a Poisson random variable with parameter λ_B (at the Academy) and λ_G (at the Learning Generation). The van carries these students back to Podunk, where they enter the post office and become the only customers of the day.

In the questions, assume that $0 < p < 1$, and that λ_B and λ_G are both strictly positive. Let the random variable X be the number of boys visiting the post office in Podunk on a given day, and Y the number of girls.

a) Calculate the probability mass function

$$p_{XY}(i, j) = P(X = i \text{ and } Y = j).$$

b) Find formulas (with no infinite sums) for the probability mass functions

$$p_X(i) = P(X = i), \qquad p_Y(j) = P(Y = j).$$

c) Prove carefully that X and Y are *not* independent. This means writing down specific subsets A and B of the real numbers and checking that

$$P(X \in A \text{ and } Y \in B) \neq P(X \in A)P(Y \in B).$$

d) Calculate $E(X)$, $E(Y)$, and $E(X + Y)$.

e) Find the probability mass function for $X + Y$,

$$p_{X+Y}(m) = P(X + Y = m).$$

If $\lambda_B = \lambda_G = \lambda$, show that $X + Y$ is a Poisson random variable with parameter λ.

f) Comment on Example 2b on page 249 in the text.

3. (20 points) Suppose we have a floor (like much of MIT) marked with parallel lines spaced 9 inches apart. We take a 6 inch pencil, toss it in the air, and note whether it lands across one of the lines, recording success if it does and failure if it does not. According to Example 2d, the probability of success is $4/3\pi$.

a) Suppose you repeat the experiment above 1000 times, achieving N successes. What value of π is suggested by this procedure?

b) How large an error (in the suggested value of π) might you expect the procedure (1000 tosses of the pencil) to have? (The question is vaguely phrased; part of the problem is to make it more precise so that you can answer it.)

c) How many times should you repeat the experiment in order to calculate π to three decimal places (that is, with an error that's probably not more than .0005)?