18.440 Problem Set 7

Due in class Monday October 27; late work will not be accepted. You can discuss problems with anyone, but you should write solutions entirely on your own.

1. (40 points) Suppose you shoot at a disc of radius one, with shots uniformly distributed over the disc. Think of the sample space as

$$S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\},$$

and the probability of $E \subset S$ as (area of E)$/\pi$. Let R be the random variable giving the distance to the center of the target: $R(x, y) = \sqrt{x^2 + y^2}$.

a) Calculate the expected value of R as an integral over S. (Hint: it isn't necessary to calculate any very unpleasant integrals.)

b) Calculate the probability density function f_R.

c) Calculate the expectation of R^2 as an integral from 0 to 1 using the probability density function.

d) Explain why the answer to (a) should be more than 1/2.

2. (40 points) Suppose X is a random variable with continuous density function f_X, that X always takes values greater than or equal to a, and that the expected value of X exists. This last assumption means that

$$E(X) = \int_a^\infty x f_X(x)\, dx = \lim_{b \to \infty} \int_a^b x f_X(x)\, dx$$

exists as an improper integral. Write

$$F_X(x) = \int_a^x f_X(t)\, dt$$

for the cumulative distribution function of X.

a) Prove that $\lim_{b \to \infty} b(1 - F_X(b)) = 0$. (This isn't difficult, but it's conceptually a bit subtle. It's the hardest part of the problem, and you should be able to do the rest even if this doesn't make sense.)

b) Prove that for any $b \geq a$,

$$\int_a^b x f_X(x)\, dx = \int_a^b (1 - F_X(x))\, dx + (-b(1 - F_X(b)) + a(1 - F_X(a))).$$

c) Prove that

$$E(X) = \int_a^\infty (1 - F_X(x))\, dx + a.$$

(In particular, you should explain why the improper integral converges.)

d) If we pick a different number $a' < a$, we get a different formula

$$E(X) = \int_{a'}^{\infty} (1 - F_X(x))\, dx + a'.$$

Explain how these two different formulas can both be correct.

3. (10 points) A fair coin is tossed 900 times. Use the normal approximation and the table on page 203 to estimate the probability that the number of heads is between 440 and 460.

4. (10 points) An exponential random variable X is "memoryless" in the sense that

$$P(X \geq s + t \mid X \geq t) = P(X > s) \qquad (s, t > 0).$$

If X represents the lifetime of some component, then this formula means that for any component that's still working, the expected future lifetime is the same. Give an example of a continuous random variable Y (taking non-negative values) with the property that

$$P(Y \geq s + t \mid Y \geq t) > P(X > s) \qquad (s, t > 0).$$

This says (in the lifetime interpretation) that old components that are still operating are likely to last longer than new ones.