18.440 Problem Set 6

Due in class Monday October 20; late work will not be accepted. You can discuss problems with anyone, but you should write solutions entirely on your own.

1. (50 points) Start with a stick of length equal to 1. Pick two numbers (x, y) between 0 and 1, independent and uniformly distributed. You should think of the sample space as being the unit square in the plane,

$$S = \{(x, y) \mid 0 \le x \le 1, \quad 0 \le y \le 1\}.$$

"Uniformly distributed" means that the probability of a subset $E \subset S$ is equal to its area. Now cut the stick at the points x and y, getting three shorter sticks. For example, if $0 \le y \le x$, then these three shorter sticks will have lengths

$$L_A(x, y) = y, \quad L_B(x, y) = x - y, \quad L_C(x, y) = 1 - x.$$

Let X be the length of the longest of the three sticks: $X = \max(L_A, L_B, L_C)$. The goal of this problem is to find the expected value of X. What makes the problem painful is the fact the formula for X depends on whether x is bigger than y, and then on the relative sizes of the three numbers L_A, L_B, and L_C. To simplify life a bit, we'll just look at the case $y \le x$; that is, we'll pretend the sample space is the right triangle

$$S' = \{(x, y) \mid 0 \le y \le x \le 1\},$$

and that the probability of an event $E' \subset S'$ is equal to TWICE its area. (This amounts to "conditioning" S on the event $y \le x$, which has probability $1/2$.)

a) The sample space S' can be divided into six regions corresponding to the six possible arrangements of the lengths:

$$R_{BAC} = \{(x, y) \in S' | L_B(x, y) \le L_A(x, y) \le L_C(x, y)\},$$

$$R_{CBA} = \{(x, y) \in S' | L_C(x, y) \le L_B(x, y) \le L_A(x, y)\}$$

and so on. Make a picture showing these six regions.

b) Write a formula for the random variable X on each of the six regions. (For example, on the region R_{CBA} the piece of length L_A is longest, so $X(x, y) = y$ on R_{CBA}.)

c) Find the probability (that is, twice the area) of each of the six regions.

d) Calculate the expected value

$$E(X) = 2 \int \int_{S'} X(x, y)\, dxdy$$

by writing the integral as a sum of the integrals over the six regions, and calculating each summand (like $2 \int \int_{R_{CBA}} y\, dxdy$) using 18.02 methods.

GO SOX

2. (20 points) Problem 12 on page 229 of the text.

3. (30 points) In the setting of Problem 12 on page 229, suppose that $0 \le p \le q \le r \le 100$. Suppose that the three service stations are placed p miles from city A, q miles from city A, and r miles from city A.

a) Write a formula (in terms of p, q, and r) for the expected distance the bus would need to be towed to the nearest service station, in the event of breakdowns uniformly distributed between cities A and B.

b) Where should the service stations *really* be located? (Hint: you're being asked to minimize the function computed in part (a), subject to the constraints $0 \le p \le q \le r \le 100$. You can use methods from multivariable calculus, or you can think of first fixing q and r, and choosing p to minimize the function. Then think of fixing q, and choosing r to minimize the function. You're finally left with a function of the single variable q to minimize.)