18.440 Problem Set 4

Due in class Monday September 29; late work will not be accepted. You can discuss problems with anyone, but you should write solutions entirely on your own.

1. (30 points) The series $\sum_{n=1}^{\infty} 1/n^2$ converges to a positive number A (actually to $\pi^2/6$, but that doesn't matter for us). The set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ can be made into a sample space by defining $P(\{n\}) = \frac{1}{An^2}$. More generally, for any event $E \subset \mathbb{N}$, define

$$P(E) = \sum_{n \in E} \frac{1}{An^2}.$$

a) With this probability on $\mathbb{N}$, show that the chance that a natural number is divisible by k is equal to $1/k^2$.
b) Find a different probability P' on the same sample space $\mathbb{N}$ so that the chance of being divisible by k is equal to $1/k^3$.
c) Is there a probability that makes the chance of being divisible by k equal to $1/k$ for every positive integer k?

2. (20 points) Let X be the number of heads appearing in eight tosses of a fair coin. Compute (from first principles, not by using the general answers found in section 4.6) the expectation and the variance of the random variable X.

3. (30 points) Problem 15 on page 181 (concerning the probability of getting an even number of heads. You can follow the hint in the book if you like, but I would suggest that it's easier to proceed by induction on n, and to condition on the outcome of the first toss.)

4. (20 points) Problem 30 on page 175 (the St. Petersburg paradox). (I assign this problem at the insistence of my son, who graduated from college a couple of years ago and therefore still knows everything. When he learned that I was teaching probability, he immediately mentioned the St. Petersburg paradox. He seemed indifferent to my response that the love of money was the root of all evil, an understandable attitude given his employment at a bank.)