18.104, Solutions to third problem set

Problem 1. This asked for an algorithm like the one in problem 7 on page 10 for computing $2^{1/3}$. I'll follow the hints pretty closely. The idea was to construct a ladder in which the nth row consists of three integers (a_n, b_n, c_n). The $(n+1)$st row was to be constructed from the nth using a three by three integer matrix A, according to the rule

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = A \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}. \tag{1}$$

The starting row is not too important; I'll take $(1, 1, 1)$ to be definite. You want to find

$$\lim_{n\to\infty} b_n/a_n = \lim_{n\to\infty} c_n/b_n = 2^{1/3}. \tag{2}$$

So the question is how to choose A cleverly so that this is true. I don't know an entirely straightforward way to proceed. One clever idea is to think that

$$r_0 = \begin{pmatrix} 1 \\ 2^{1/3} \\ 2^{2/3} \end{pmatrix}$$

should be an eigenvector for A (since the line through this vector is the only one having exactly the properties we are trying to approximate). Then you are trying to find a matrix with integer entries for which r_0 is an eigenvector. Again this is a hard problem, and I don't know a really easy way to proceed. If you just cyclically permute the coordinates of r_0, then you get something very close to $2^{1/3} r_0$. Expressed using matrices, this says that if

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix},$$

then $A_0 r_0 = 2^{1/3} r_0$.

Now we're making some progress: a reasonable first guess for our ladder has nth row equal to (the transpose of) $A_0^n (1, 1, 1)^t$. So what happens with the property (2) that we're trying to achieve? Well, we want to understand what happens when we apply A_0 repeatedly to some vector. A good way to do that is using eigenvectors. It's not hard to see (since $A_0^3 = 2I$) that the eigenvalues of A_0 are

$$q_0 = 2^{1/3}, \quad q_+ = 2^{1/3}\omega, \quad q_- = 2^{1/3}\omega^{-1}.$$

Here $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$ is a complex cube root of one. We already know the eigenvector r_0 for $2^{1/3}$; the other two eigenvectors are

$$r_\pm = \begin{pmatrix} 1 \\ \omega^{\pm 1} 2^{1/3} \\ \omega^{\pm 2} 2^{2/3} \end{pmatrix}.$$

Now we know that

$$A_0^n r_i = q_i^n r_i \qquad (i = 0, +, -).$$

We can therefore understand our ladder in the following way. Express the initial row in terms of the eigenvectors as

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = u_0 r_0 + u_+ r_+ + u_- r_-,$$

with u_0 and $u_\pm$ complex numbers. Then

$$A_0^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = u_0 q_0^n r_0 + u_+ q_+^n r_+ + u_- q_-^n r_-.$$

We're interested in the ratios of coordinates in the nth row. The ratios will be unchanged if we divide the vector by $u_0 q_0^n$, giving
$$r_0 + (u_+/u_0)(q_+/q_0)^n r_+ + (u_-/u_0)(q_-/q_0)^n r_-.$$
Using the formulas above for the eigenvalues, this becomes
$$r_0 + (u_+/u_0)\omega^n r_+ + (u_-/u_0)\omega^{-n} r_-.$$
The first term here has the ratios we want, so we would like to know that the second and third terms go to zero. Unfortunately this is not true: the complex number ω has absolute value one, so the coordinates of the second and third terms don't change in absolute value. They do not go to zero, and we haven't achieved property (2). (Actually a little experimentation would have made this failure obvious. The rows of the ladder here are
$$(1,1,1), \quad (1,1,2), \quad (1,2,2), \quad (2,2,2), \quad (2,2,4), \quad (2,4,4), \quad (4,4,4), \ldots$$
and so on: they simply double every three steps.)

Fixing this problem requires another clever idea. What's wrong with A_0 is that the eigenvalue we want (the cube root of 2) has exactly the same absolute value as the two we don't want ($\omega^{\pm 1} 2^{1/3}$). Suppose we replace A_0 by
$$A = A_0 + I = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}. \tag{3}$$
The eigenvectors are unchanged, but the eigenvalues of A are
$$p_0 = 2^{1/3} + 1, \qquad p_+ = \omega 2^{1/3} + 1, \qquad p_- = \omega^{-1} 2^{1/3} + 1. \tag{4}$$
Arguing as for A_0, we find that the nth row of our new ladder is
$$A^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = u_0 p_0^n (r_0 + (u_+/u_0)(p_+/p_0)^n r_+ + (u_-/u_0)(p_-/p_0)^n r_-). \tag{5}$$
What makes this formula work for us is the fact that $|p_\pm/p_0| < 1$ (an easy calculation), and therefore that
$$\lim_{n\to\infty} (p_\pm/p_0)^n = 0.$$
Now (2) follows easily.

2. You were given two estimates
$$|\cos y - (1 - y^2/2)| \le E y^4, \qquad |y| \le 1, \tag{6}$$
$$|\log(1+z) - z| \le F z^2, \qquad |z| \le 1/2. \tag{7}$$
(I wrote strict inequalities in the estimates, but then they're false for y or z equal to zero.) You were to compute
$$\lim_{n\to\infty} (\cos(x/\sqrt{n}))^n.$$
A natural way to proceed is to try to compute the logarithms of the terms in the sequence. Notice that if $|x| < \sqrt{n}$, then
$$\cos(x/\sqrt{n}) > \cos(1) > \cos(60^\circ) = 1/2. \tag{8}$$
So we're talking about logarithms of positive numbers, and the approximation (7) can be used for the log of the cosine. Let's write $e(y)$ and $f(z)$ for the errors in (6) and (7), so that
$$\cos(y) = 1 - y^2/2 + e(y), \qquad \log(1+z) = z + f(z). \tag{9}$$

Then (6) and (7) provide information about how big e and f are:

$$|e(y)| \leq Ey^4, \qquad |y| \leq 1, \tag{10}$$

$$|f(z)| \leq Fz^2, \qquad |z| \leq 1/2. \tag{11}$$

We now calculate (assuming $x < \sqrt{n}$)

$$\begin{aligned}
\log((\cos(x/\sqrt{n}))^n) &= n\log(\cos(x/\sqrt{n})) = n\log(1 - x^2/2n + e(x/\sqrt{n})) \\
&= n(-x^2/n + e(x/\sqrt{n}) + f(x^2/2n + e(x/\sqrt{n})) \\
&= -x^2 + ne(x/\sqrt{n}) + nf(x^2/2n + e(x/\sqrt{n}))
\end{aligned}$$

.

The last two terms are errors that we want to estimate. Because we are assuming $x/sqrtn < 1$, we can use the estimate (10) for e:

$$ne(x/\sqrt{n}) \leq nE(x^4/n^2) = Ex^4/n.$$

The inequality in (8) guarantees that the argument of f is at most 1/2 (think about this!) so (11) gives

$$f(x^2/2n + e(x/\sqrt{n})) \leq F(x^2/2n + Ex^4/n)^2.$$

The sum of the two error terms is therefore at most

$$Ex^4/n + F(x^2/2n + Ex^4/n)^2.$$

This tends to zero as $n \to \infty$ (since x is fixed), proving that

$$\lim_{n\to\infty} \log(\cos(x/\sqrt{n}))^n = -x^2/2.$$

Since the exponential function is continuous, we can apply it to both sides of this, and conclude that

$$\lim_{n\to\infty} (\cos(x/\sqrt{n}))^n = \exp(-x^2/2).$$

page 264, # 13. The hint in the text is pretty helpful, so I'm not going to write this one out.

Possible topics for future presentations

For 1.5 hour talks (with lecture notes, etc):

Book, section 2.1: elementary prime number theory. Might include looking at problems 1, 2, 14, or 15 on pages 70–71.

Book, section 5.5: fancier prime number theory. Might include problems 1, 2, 4, 11, or 12.

Fibonacci numbers and the golden mean: book, sections 3.1 and 3.2.

Book, section 3.3: generating functions.

Defining $n!$ for non-integer n: see the book *The Gamma Function* by Emil Artin.

How computers calculate trigonometric and exponential functions. (I know almost nothing about this; you'd need to find your own sources.)

Weyl's theorem: if x is any irrational number, then the exponentials parts $\exp(2\pi i n x)$ are uniformly distributed over the unit circle. (Proof is in some introductory books on Fourier series.)

For short presentations next week:

Any past homework problems.

Book, pages 72–73; problems 16–18. (Finite differences).

Book, problems about Fibonacci numbers, pages 134–140.