18.104, Solutions to first problem set

Problem 1. Writing (a_n, b_n) for the nth row of the ladder, you had no difficulty noticing the rules

$$a_{n+1} = a_n + b_n, \qquad b_{n+1} = 2a_n + b_n, \tag{1.1}$$

and proving by induction on n that

$$b_n^2 - 2a_n^2 = (-1)^n. \tag{1.2}$$

Part (c) asked you to prove that

$$\lim_{n\to\infty} b_n/a_n = \sqrt{2}, \tag{1.3}$$

and that the errors

$$\epsilon_n = \sqrt{2} - b_n/a_n \tag{1.4}$$

decrease in absolute value with n. This caused a bit of trouble. From (1.2) you get immediately

$$\left(\frac{b_n^2}{a_n^2}\right) - 2 = (-1)^n/a_n^2. \tag{1.5}$$

The difficulty is that this equation does not directly tell you about ϵ_n. (To deduce the convergence, you need to know that b_n/a_n is always positive (which is clear), that $a_n \to \infty$ (in fact it's not hard to show by induction that $b_n \geq a_n \geq 2a_{n-1}$, and therefore that $a_n \geq 2^{n-1}$), and that the square root function is continuous.)

Here is a one way to show that the approximations actually improve constantly. Since $b_n/a_n > 0$ for all n, and since the function x^2 is strictly increasing for positive x, equation (1.5) implies that the sequence b_{2m+1}/a_{2m+1} is increasing, and b_{2m}/a_{2m} is decreasing. So all of these quotients are between 1 and 3/2. In that interval, the derivative of x^2 is between 2 and 3. By the mean value theorem (applied to x^2 on the interval between b_n/a_n and $\sqrt{2}$), it follows that

$$\frac{1}{3a_{2m+1}^2} < \epsilon_{2m+1} < \frac{1}{2a_{2m+1}^2} \tag{1.6}$$

and that

$$\frac{1}{3a_{2m}^2} < -\epsilon_{2m} < \frac{1}{2a_{2m}^2}. \tag{1.7}$$

(These inequalities make it obvious that the errors converge to zero.) To prove that the errors are decreasing, it's enough to show that the lower bound for the nth error is smaller than the upper bound for the $n+1$st; that is, that

$$\frac{1}{3a_n^2} < \frac{1}{2a_{n+1}^2},$$

or equivalently

$$a_{n+1}^2 > \frac{3}{2}a_n^2.$$

But I already remarked that the recursion formula gives $a_{n+1} \geq 2a_n$; squaring this implies the inequality needed.

Problem 2. The statement in part (a) is not correct. Take $a = 1$, and the approximation $x_1 = 1/4$; the error is $3/4$. Then $x_2 = (1/4 + 4)/2 = 17/8$, an error of $9/8$. What is true is that all the approximations *after* the first are greater than $\sqrt{a}$, and that they decrease.

There are various ways to prove this algebraically, but none of you found a way to involve the geometry of the hyperbola $xy = a$. So suppose x_n is positive, and not equal to $\sqrt{a}$. Then $(x_n, a/x_n)$ and $(a/x_n, x_n)$ are two points on the hyperbola. If you draw the line segment connecting these two points, then the midpoint is (x_{n+1}, x_{n+1}). (Why?) Because the hyperbola is convex (the second derivative of the function a/x is positive), the line segment lies above the hyperbola. This means that $x_{n+1} > a/x_{n+1}$, or equivalently $x_{n+1} > \sqrt{a}$. (So we know that all the approximations after the first are greater than $\sqrt{a}$.)

If also $x_n > \sqrt{a}$ (for example if n is at least 2), then $a/x_n < \sqrt{a} < x_n$, so the midpoint x_{n+1} of the interval $[a/x_n, x_n]$ is less than x_n, as we wished to show.

For part (b), we may as well assume that n is at least 2, so that

$$\epsilon_n = x_n - \sqrt{a} > 0. \tag{2.1}$$

Now let's try to estimate ϵ_{n+1}:

$$\begin{aligned}\epsilon_{n+1} &= x_{n+1} - \sqrt{a} = (x_n + a/x_n)/2 - \sqrt{a} \\ &= (x_n - \sqrt{a})/2 + (a/x_n - \sqrt{a})/2. \\ &= \epsilon_n/2 + (a/x_n - \sqrt{a})/2.\end{aligned}$$

The second term in the last equation is negative (since we are assuming $x_n > \sqrt{a}$, so we get

$$\epsilon_{n+1} < \epsilon_n/2.$$

Since we already know that ϵ_{n+1} is positive, this proves that the errors are at least halved.

You had no trouble with (c). For (d), most of you discovered the formula

$$x_n = b_{2^{n-1}}/a_{2^{n-1}} \tag{2.2}$$

relating Heron's approximations to $\sqrt{2}$ (with $x_1 = 1$) to those of Problem 1, but nobody gave a complete proof. The difficulty is that although you know a simple relation between (a_n, b_n) and (a_{n+1}, b_{n+1}), you *don't* know a simple relation between (a_{2^m}, b_{2^m}) and $(a_{2^{m+1}}, b_{2^{m+1}})$. To get one, you need a slightly more sophisticated way to think about the ladder. Look at the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}. \tag{2.3}$$

If you think of the rows of the ladder as vectors in $\mathbb{R}^2$, then the recursion formula is

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = A \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.4}$$

It's also easy to prove that

$$\begin{pmatrix} b_n \\ 2a_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Combining these two formulas gives

$$A^n = \begin{pmatrix} b_n & a_n \\ 2a_n & b_n \end{pmatrix}.$$

Now the formula $A^{2n} = (A^n)^2$ gives by matrix multiplication a recursion formula

$$b_{2n} = b_n^2 + 2a_n^2, \qquad a_{2n} = 2a_n b_n.$$

On the other hand, take b/a as an approximation to $\sqrt{2}$. Then Heron's formula gives as the next approximation

$$(b/a + 2a/b)/2 = (b^2 + 2a^2)/2ab.$$

Comparing the last two formulas shows that if $x_m = b_n/a_n$, then $x_{m+1} = b_{2n}/a_{2n}$. Since we are taking $x_1 = 1 = b_1/a_1$, the desired formula (2.2) follows by induction.

Problem 3 asked about similar results for $\sqrt{m}$. Of course most of Problem 2 applies to arbitrary square roots, so this is first of all a question about generalizing Problem 1. One way is to use a ladder based on the relations

$$a_{n+1} = a_n + b_n, \qquad b_{n+1} = ma_n + b_n, \tag{3.1}$$

beginning again with $a_1 = b_1 = 1$. This ladder is related to the matrix

$$A_m = \begin{pmatrix} 1 & 1 \\ m & 1 \end{pmatrix};$$

arguing as in Problem 2, you can show that

$$A_m^n = \begin{pmatrix} b_n & a_n \\ ma_n & b_n \end{pmatrix}. \tag{3.2}$$

Taking determinants gives a relation like (1.2), but not so nice:

$$b_n^2 - ma_n^2 = (1-m)^n$$

Hence

$$\left(\frac{b_n^2}{a_n^2}\right) - m = \frac{(1-m)^n}{a_n^2}. \tag{3.3}$$

In order to deduce that the sequence b_n/a_n converges to $\sqrt{m}$, we need to show that the left side goes to zero; that is, we need to know that a_n is fairly large. The recursion relations (3.1) lead to easy inductive proofs (assuming $m \geq 1$) that

$$b_n \geq a_n \geq b_{n-1}$$
$$a_{n+2} \geq (m+3)a_n$$
$$a_n^2 \geq (m+3)^{n-2}.$$

Inserting the last of these in (3.3) gives

$$|\left(\frac{b_n^2}{a_n^2}\right) - m| \leq (m-1)^2 \left(\frac{m-1}{m+3}\right)^{n-2}. \tag{3.4}$$

The right side converges to zero as $n \to \infty$, so the ratios b_n/a_n converge to $\sqrt{m}$.

Squaring (3.2) gives recursion formulas

$$a_{2n} = 2a_n b_n, \qquad b_{2n} = b_n^2 + ma_n^2. \tag{3.5}$$

If we use Heron's formula beginning with $x_1 = 1/1$ to form a sequence $x_k = p_k/q_k$ of approximations to $\sqrt{m}$, then

$$x_{k+1} = \frac{1}{2}\left(\frac{p_k}{q_k} + \frac{mq_k}{p_k}\right) = \frac{p_k^2 + mq_k^2}{2p_k q_k}. \tag{3.6}$$

Comparing (3.5) with (3.6) shows that

$$x_k = b_{2^{k-1}}/a_{2^{k-1}},$$

just as in Problem 2.

Problem 4. This concerned the function $r(N)$ counting integer solutions of $a^2 + b^2 = N$. The problem was to find large values of $r(N)$ with small values of N. Most of you gave a table of such large values, typically extending to $r(71825) = 72$; but almost nobody said much after that.

One way to talk about this problem is to introduce the function

$$R(x) = \max_{0 \le N \le x} r(N),$$

and to try to estimate $R(x)$. I showed in class that $r(N) \le 4\sqrt{N} + 1$, and it follows at once that $R(x) \le 4\sqrt{x} + 1$. Finding lots of large values of r means finding *lower* bounds for the function R. If $N = 5^m$, then $r(N) = 4(m + 1)$. The largest power of 5 less than or equal to x is $[\log 5(x)]$. (Here $[t]$ is the largest integer less than or equal to t.) It follows that

$$R(x) \ge 4([\log_5(x)] + 1) \ge 4\log_5(x) = \left(\frac{4}{\log 5}\right)\log(x).$$

Using powers of 65 in a similar way gives

$$R(x) \ge \left(\frac{4}{\log 65}\right)\log^2(x).$$

For any positive integer k you can find k distinct primes $p_1,\ p_2,\ldots p_k$ congruent to 1 mod 4. (This is not obvious, but it's true.) Using powers of their product gives

$$R(x) \ge \left(\frac{4}{\log p_1 + \cdots + \log p_k}\right)\log^k(x).$$

That is, $R(x)$ grows faster than any power of $\log x$. This is still a very long way from the upper bound $4\sqrt{x} + 1$

To do much better requires first of all a deeper knowledge of the sequence of primes congruent to 1 mod 4. I won't try to do that here.

Homework for Thursday, September 27

1. Page 95, # 2.
2. Page 97, # 11.
3. Page 99, # 19.
4. Find a closed formula (better than the one in (2.4) above) for the entries a_n and b_n of the ladder on page 10, problem 7.