18.100B Problem Set 6

Due in class Monday, April 8. You may discuss the problems with other students, but you should write solutions entirely on your own.

1. Give a proof or a counterexample for the following assertion: suppose that f is a real-valued differentiable function on the interval (-1,1), and that $f'(0) = 1$. Then there is an $\epsilon > 0$ so that f is strictly increasing on the interval $(-\epsilon, \epsilon)$.

2. Text, page 114, number 4.

3. Text, page 115, number 9.

4. For this problem, you may assume that for every non-negative integer N,

$$\lim_{x \to 0} |x|^{-N} \exp(-1/|x|) = 0.$$

(This is the same statement as $\lim_{t \to \infty} t^N \exp(-t) = 0$, which may look a bit more familiar.)

The problem concerns the function

$$f(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

a) Sketch the graph of f.

b) Show that for each $k \geq 0$ there is a polynomial $p_k(x)$ so that the kth derivative of f is

$$f^{(k)}(x) = \begin{cases} \left(p_k(x)/x^{2k}\right) \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

c) Show that $f^{(k)}(0)$ exists for all $k \geq 0$.

d) For which values of x is $f(x)$ equal to the sum of the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k?$$