18.100B Problem Set 4

Due in class Monday, March 11. You may discuss the problems with other students, but you should write solutions entirely on your own.

1. Suppose $\{p_n\}$ and $\{q_n\}$ are two sequences in a metric space X, that $\{p_n\}$ converges to p, and that $\{q_n\}$ converges to q. Prove that

$$\lim_{n\to\infty} d(p_n, q_n) = d(p, q).$$

2. Suppose that $\{x_n\}$ is a sequence of non-negative real numbers converging to a non-negative real number x. Prove that

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}.$$

(Hint: it may help to consider separately the cases $x = 0$ and $x \neq 0$.)

3. Find a sequence of real numbers for which every positive integer is a subsequential limit.

4. This problem concerns the 2-adic metric on $\mathbb{Z}$, defined by

$$d_2(m,n) = \begin{cases} 0, & \text{if } m = n \\ 2^{-a}, & \text{if } m \neq n \text{ and } 2^a \text{ is the largest power of 2 dividing } m - n. \end{cases}$$

Let X be the metric space $\mathbb{Z}$ with the 2-adic distance. Show that the sequence $\{1, 3, 7, 15, 31, \dots\}$ in X converges. (This can be phrased in terms of infinite series as saying that the series

$$1 + 2 + 4 + 8 + \cdots = \sum_{j=0}^{\infty} 2^j$$

converges in X. You may know a formula for summing such a series that will help you guess the limit of the sequence.)

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